PROCEEDINGS

OF THE

NATIONAL ACADEMY OF SCIENCES

Volume 26 November 15, 1940 Number 11

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*THE INFLUENCE OF NERVE FIBERS UPON TASTE BUDS DURING EMBRYONIC DEVELOPMENT**

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Communicated September 30, 1940

It has long since been demonstrated in a variety of animal forms that subsequent to experimentally induced degeneration of gustatory nerve fibers the taste buds associated with them disappear. It has also been shown that when the nerve fibers regenerate, new buds make their appearance. Taste buds, therefore, are intimately dependent for their existence upon their nervous supply; and by way of explaining the possible physiological nature of this dependence, a neurohumoral mediator has been hypothesized (Torrey, '34*a*, '36).

On the basis of accounts given by Hermann ('84), Gräberg ('98) and others, it has been inferred that nerves probably play a similar rôle in the ontogeny of gustatory organs. Hermann, for instance, maintained that nerve fibers came into contact with the epithelium of the developing papillae on the tongue of the foetal rabbit at the same time the anlagen of the taste buds were appearing, and thus nerves were presumed to have stimulated or induced the formation of the sense organs. Szymonowicz ('95, '96) made a similar suggestion for the Corpuscles of Grandry and Herbst, and more recently Whiteside ('26), speaking of the regeneration of taste buds in the rat, has said that inasmuch "as the regenerative process resembles the embryonic, the causative factor would probably be the same in both cases."

Other workers, by contrast, have not accepted this inference. Herbst ('01) and Harrison ('04), for example, have said that the coincidental appearance of nerve fiber and sense organ is not proof of the relation of cause and effect. Harrison has gone further and like Roux differentiated between a "function of development" and a "function of conservation;" that is, the power to differentiate is inherent within the anlage itself, whereas the stimulus that subsequently maintains the organ is supplied by the nerve.

This conception is furthered by Patzelt ('24) who maintains that not only is the ability to differentiate into taste buds inherent within the epithelium, but that the nerve fibers actually grow in secondarily. And finally, Stone ('33, '40) concludes from his studies of grafted tongue primordia in Amblystoma that in this form, at least, nerves do not play a causative rôle in the formation of taste buds.

Obviously the intimate dependence of taste buds upon their nerve supply exhibited during degeneration and regeneration in the adult individual has not been demonstrated to exist for certain during ontogeny. An attempt has been made, therefore, to solve this long-time problem. The investigation has followed two courses. The first consisted of working out the normal embryogeny of the sense organs and their nerve fibers. This was necessarily prerequisite to the second: an experimental analysis of the morphogenetic association prevailing between the two. All experiments and observations have been performed on the rat and attention directed solely towards the vallate papilla of which there is a single one only in the rat.

The development of the vallate papilla (as of all the other papillae) precedes that of the taste buds by a considerable time. The single papilla originates in a manner so similar to that already described for those of the rabbit (Hermann, '84) and man (Gräberg, '98), that additional description is hardly warranted. Suffice it to say for the rat that the papilla begins to take form towards the end of the fifteenth day of gestation and is completely established by the third day postpartum. The first taste buds, however, do not appear until about the ninth day after birth. This cannot be said to represent an absolute date, for some individual variation occurs. In one or two instances, for example, an occasional bud was to be seen early on the eighth day, whereas on other occasions buds could not be detected until the tenth. Generally speaking, however, the time of initial appearance of the buds is the ninth day. Few in number at this time, they gradually multiply, reaching a maximum in about twelve weeks, from which time they undergo a slight reduction in numbers until an adult constant of roughly 75 to 100 buds, associated exclusively with ventral levels of the papilla and bounding trench, is attained.

Special attention was directed towards the time of arrival of the ingrowing nerve fibers at their epithelial terminals. For this purpose whole embryos and pieces of tongues ranging from a prenatal age of fifteen days to twelve days postpartum were fixed in formol-acetic-alcohol and stained with activated protargol (Bodian, '37).

Other workers have shown that the single anlage of the glossopharyngeal ganglia is established towards the beginning of the twelfth day of gestation and promptly divides into a dorsal group of cells, the future superior ganglion, and a ventral group, the future petrosal ganglion. Processes are soon

given off from the neuroblasts and by the end of the thirteenth day a definitive glossopharyngeal nerve has connected centrally with the brain and is pushing peripherally towards its final terminations.

My own observations reveal that as soon thereafter as the beginning of the sixteenth day a fairly conspicuous bundle of fibers is present in the basal portions of the developing tongue lying not far beneath the dorsal epithelium. These fibers continue peripherally during the succeeding two days, reaching and even extending beyond the level of the already developing vallate papilla. Sections of a twenty-day tongue reveal nerve fibers which have entered and traversed the subepithelial tissues of the papilla to terminate in the form of a subepithelial plexus. From this time until the first taste buds appear some ten days later the number of fibers within the papilla, and thus at the same time the extent of the subepithelial plexus, are increased by the arrival of new bundles of fibers from various directions beyond the papilla.

A few of the preparations of postpartum stages previous to the ninth day show occasional fibers which have penetrated the papillary epithelium. Such intraepithelial fibers are so difficult to demonstrate histologically that an accurate count of their absolute number has not been attempted. Sufficient of them are stained, however, to demonstrate unquestionably that such *fibers are present several days in advance of taste bud differentiation.*

If one is committed to the proposition that nerves do incite or determine the formation of taste buds, then the above facts lend themselves in support. The precocious arrival of the fibrils at the site of the future end organs may, however, be entirely without significance other than that a nervous supply is ready and waiting for the buds when they do appear, for it is yet to be demonstrated that the fibers literally exert a "stimulus of development" which brings the end organs into existence. A critical test would consist in setting up conditions whereby the buds would be permitted, if they are capable of doing so, to develop free from the influence of nervous elements. This has been attempted in two entirely different fashions: first, by causing a previously destroyed papilla in the adult to regenerate in the absence of functional nerve fibers; second, by transplantation of papillae of pre-taste bud ages.

Whiteside ('26) has shown that following complete extirpation, the vallate papilla of the rat begins to regenerate within three weeks and in six months is completely restored. New taste buds likewise develop in the epithelium of the regenerating papilla, the first buds usually appearing about the same time the papilla proper begins to take form, i.e., at three weeks, and continuing to be formed for as long as twenty-two weeks.

In the light of the known relations of gustatory organs to degenerating and regenerating nerves, buds are formed out of the newly established epithelium of a regenerating papilla presumably under the influence of the

gustatory fibers likewise resupplying the papilla. Assuming that the regeneration process is closely akin to the original ontogeny, a test of the taste bud forming capabilities of a regenerating papilla *devoid* of functional nerves might be expected to give a partial answer to the question of the ontogenetic relation of the sense organ and nervous supply. The conditions for such a test have been set up by destroying the vallate papilla and cutting the glossopharyngeal nerves which supply it.

Four to seven months old male and female albinos were used for most of these experiments, supplemented by a few individuals from a chocolate hooded stock employed for the transplantation experiments to be described later. Individuals were anesthetized with sodium amytal and the vallate papilla cauterized. Because most of the buds have a bilateral innervation (Whiteside, '27) it was necessary to cut both glossopharyngeal nerves. Each was located where it lay dorsal to the anterior belly of the digastricus muscle and severed proximal to the junction of its pharyngeal and lingual branches.

Fifteen animals were operated upon in this fashion. For controls fifteen animals with the papillae alone destroyed were used. Three experimental and three control animals were sacrificed two weeks after operation and the tongues prepared for study. Additional groups of three each were sacrificed so as to comprise a series with the groups separated by four-day intervals. There was thus available material of fourteen, eighteen, twenty-two, twenty-six and thirty days post-operative ages.

At the fourteen-day stage in both the experimental and control animals, the regeneration process has gone little beyond the wound healing stage. The papilla has not yet begun to reform and there are no taste buds present. Four days later there are evidences of the epithelial proliferation preceding the formation of a definite papilla. The papilla gradually begins to take form during the succeeding stages in both the experimental and control animals. The details of its regeneration conform to those already given by Whiteside. The one additional contribution made by the present investigation is that regeneration of the papilla goes on independent of the nervous system; the known dependence of regenerating taste buds on nerve fibers does not prevail for the papilla. It is interesting to note in this connection that the barbels of the catfish, by contrast, will not regenerate in the absence of a normal nerve (Olmsted, '20).

As far as the buds themselves are concerned, the first newly formed ones were observed in the control animals of twenty-six days. There were three buds in one of these, and five in the other two. The three controls of the thirty-day stage showed twelve buds in one, thirteen in the second and fifteen in the third. These observations on the number and time of first appearance of the buds conform fairly closely to those recorded by Whiteside.

In the experimental animals, by contrast, buds were observed in only one instance, that of one of the three thirty-day individuals, and in this case two well-defined buds were seen. There was reason to believe, however, that regenerating nerve fibers from the central trunk may have reached the papilla and induced the buds to form out of the regenerating papillary epithelium. To test this likelihood the papillae of four animals were cauterized at one time and the nerves severed ten days later. This was considered an interval sufficient to prevent the normal nerves from exerting any influence upon the reforming papilla while at the same time shortening the period available for the return of the new fibers. Two of the animals were sacrificed thirty days after cautery and two at thirty-five days. There were no taste buds present in any of the four.

The conclusion to be derived from these experiments, then, is that in the adult individual *taste organs regenerate only under the influence of nerve fibers*. To carry this conclusion over to the embryonic origin of the buds, however, requires the assumption that the regeneration of an organ is strictly comparable to its ontogeny, an assumption not wholly justified by the comparative data of regeneration phenomena in general. It remains to be demonstrated that this "function of conservation" on the part of nerves exists as a "function of development." The following experiments comprise an attempt at such a demonstration.

A crucial test of the ability of the developing tongue to give rise to taste buds in the absence of nervous stimulation involves setting up conditions whereby embryonic differentiation of the papilla would continue in an environment totally devoid of nervous elements. It was believed these conditions would be found in some variety of graft.

Preliminary experiments showed that subcutaneous, omental and ear grafts were unsuitable for several reasons. Not only did the transplanted tissues degenerate rather rapidly, but were subject to distorting pressures and were commonly invaded by considerable amounts of fat and connective tissue. There was the ever-present possibility, too, that such grafts, even if they survived and continued differentiation, would be invaded by nerve fibers from the surrounding host tissues. This eventuality alone would defeat the purpose of the experiment. It was necessary, then, to employ an environment of a more "neutral" character. To this end resort was had to the anterior chamber of the eye, a site offering many of the advantages of explanation methods and few of the technical difficulties.

The material, host and donor, for such grafts was derived from a strain of chocolate hooded rats inbred for 25, 26 and 27 generations. The original 25th generation stock was obtained through the courtesy of Dr. R. T. Hill of the Indiana University Medical School. Later in the project this material was supplemented by Sprague-Dawley albino hosts and donors which proved to be equally satisfactory. The transplants were obtained from

donors ranging from a foetal age of sixteen days to eight days postpartum (the interval during which the vallate papilla and its nervous supply develop, but terminating just short of the time of the initial appearance of taste buds). The age of the host animals ranged from 120 to 212 days.

The accompanying table summarizes the history of the grafts.

TABLE 1

EXPT. NO.	NO. OF RECOVERED GRAFTS	DONOR	AGE OF GRAFT (DAYS AFTER OPER.)	THEORETICAL POST-PARTUM AGE	AGE BEYOND NORMAL TASTE BUD THRESHOLD	TASTE BUDS
Te-1	5	1 day p. p.	10 days	11 days	2 days	None
Te-2	6	4 day p. p.	15 "	19 "	10 "	None
Te-3	4	16 day foetus	20 "	15 "	6 "	None
Te-4	5	18 day foetus	14 "	11 "	2 "	None
Te-5	8	6 day p. p.	6 "	12 "	3 "	None
Te-6	10	8 day p. p.	3 "	11 "	2 "	Present in all
Te-7	8	3 day p. p.	8 "	11 "	2 "	None
Te-8	4	20 day foetus	11 "	10 "	1 "	None
Te-9	7	7 day p. p.	5 "	12 "	3 "	Present in 3 cases

It will be observed that in all the groups of experiments the grafts were permitted to grow for a period of time in excess of that normally required for the initial appearance of taste buds. Group Te-4, for example, comprises transplants of an original foetal age of eighteen days cultivated for two weeks. Three days out of those two weeks can be thought of as belonging to the gestation period; thus the recovered graft is at a theoretical age of eleven days postpartum. It will be recalled that this age is two days beyond the average ninth-day threshold at which taste buds normally appear. Likewise the other groups of transplants, with one exception, were allowed to attain post-threshold ages of two or more days (cf. table). The exception is group Te-8 which approaches within one day of the average ninth-day threshold, but it is still within the range of observed variation in normal time of taste bud appearance.

As far as the time element alone is concerned, therefore, taste buds should have developed in all the grafts if they were capable of spontaneously doing so. On the contrary, buds appeared in only two groups of transplants; those grafted at eight (Te-6) and seven days (Te-9) of age. In the former, from two to five buds were found in all ten of the recovered grafts; buds were found in only three out of seven cases of the latter group.

This immediately suggests that determination of the taste buds, whatever the mechanism thereof, occurs between the seventh and eighth days after birth. When it is recalled that nerve fibers normally arrive at the site of future taste buds not later than the twentieth prenatal day, it appears possible that they have been involved in the determination process. However, their "function of development," if they perform such, must not

come into play before the seventh day, otherwise taste buds should have appeared in the grafts of earlier ages.

It may still be argued, of course, that the determination mechanism is inherent within the papillary epithelium or involves some other unidentified factor, and later trophic associations with the nerve fibers are secondarily assumed. But if this were so, it is difficult to understand why gustatory organs do not appear in a graft of tissue of an early age which has been successfully cultivated beyond the bud threshold and in which the papillary epithelium has otherwise undergone an apparently normal developmental history. It may be stated in connection with this last point that a definite structural regression occurred only in the longer cultivated grafts of groups Te-2 and Te-3. The papillary epithelium in the other taste bud-free grafts appeared to have otherwise made normal developmental progress.

To recapitulate: Whereas nerve fibers arrive within a developing vallate papilla not later than the twentieth prenatal day, taste buds do not appear until some ten days later. These facts maybe interpreted as favoring the idea that nerves exert a stimulus or function of development; at least they are not contradictory, although they might be interpreted in other ways. More significant is the observation that taste buds will not reappear in a regenerating papilla unless their nervous supply is intact. But this can be considered a true function of development on the part of nerves only if it is first assumed that the regeneration of an organ is homologous to its ontogeny. The final set of experiments, therefore, lends the greatest support to the concept, for it is observed that in the nerve-free environment of the anterior eye chamber taste buds fail to appear except in those papillae transplanted just previous to the time the buds normally appear. It is thus very probably true that nerve fibers do perform a function of development upon developing gustatory organs. It is recognized, however, that *the* critical experiment remains to be performed: the tongue itself of the developing animal must be left intact and undisturbed while its nerve supply in some fashion is prevented from reaching it. At the present time the technical problems barring this experiment have not been solved.

*Contribution No. 286 from the Zoölogical Laboratories, Indiana University.

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TOUCH-AND-GO PAIRING IN CHROMOSOMES

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Communicated October 8, 1940

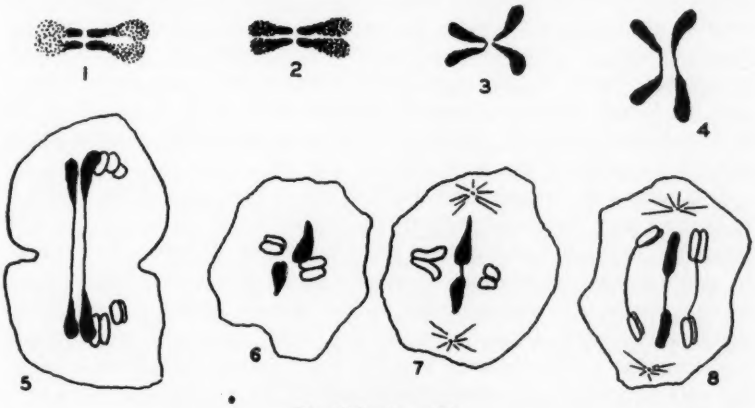
An orientation of chromosomes that appears to be independent of meiotic synapsis in the ordinary sense is shown in a relatively large number of species. Usually this takes the form of a vis-à-vis position on the spindle and when such chromosomes are separated by a considerable distance is called "distance conjugation" (Lorbeer, '34).[1] In other cases the involved chromosomes may actually come together for a period which in some instances is so brief that the movement has received the name "touch-and-go" process (Wilson, '25).[2]

The mechanism involved is a very puzzling one. In a general way the explanations that have been offered fall under two headings: (*a*) that some kind of attraction between the chromosomes is involved (Wilson, '32)[3] or (*b*) that it is mitotic forces ("centromere-spindle relationship") that bring about this orientation and that no specific attraction is involved (Darlington, '39).[4]

This latter explanation fails to account for the fact that in some instances such chromosomes approach each other before the spindle has been fully formed. Thus the *m* chromosomes in some cells of Alydus come to-

gether already in diakinesis (Reuter, '30),[5] and the XY pair of several pentatomids undergoes its brief union in the interkinesis or even the telophase prior to its disjunction. These cases seem to bespeak the existence of some kind of attraction and the following case lends additional support to such an argument.

In the male of the hemipter *Rhytidolomia senilis*—as has been pointed out elsewhere (Schrader, '40)[6]—the meiotic tetrads are formed by the terminal union of homologues in diakinesis. This is true also of the sex chromo-



Rhytidolomia senilis

Figures 1 and 2. The sex chromosomes in diakinesis. They show the equational split and have come together terminally at their euchromatic ends. The heterochromatic ends are not fully condensed. The larger X is on the right.

Figures 3 to 5. Progressive stages in the division of the sex chromosomes in the first division. Poleward movement occurs with the large of heterochromatic end foremost. The X is on the right.

Figure 6. Late telophase of interkinesis with X and Y beginning to show orientation toward each other.

Figure 7. Metaphase or early anaphase showing the large ends of X and Y oriented toward each other. Probably they are here already beginning to separate.

Figure 8. Late anaphase showing the loss of the club shape in the sex chromosomes.

somes; but it is to be noted that the X and Y chromosomes are each composed of a euchromatic and a heterochromatic section and that their diakinetic union always occurs at the euchromatic ends. This fact can be determined with little difficulty by tracing the chromosomes through the prophases. But it can also be demonstrated in the fully condensed chromosomes because both the X and the Y are clubshaped and the narrow end where the union takes place represents the euchromatic section (Figs. 1 to 3).

The first division is, superficially at least, equational for both sex chromosomes. Their movement to the poles always occurs with the large or heterochromatic end foremost. Already in the initial steps of the division all connection between the *X* and *Y* disappears (Figs. 4 and 5), but in telophase or interkinesis they become reassociated. However, in contrast to the union that takes place during diakinesis, they now come together at their large or heterochromatic ends. The approach usually begins already in late telophase when the metaphase grouping of the chromosomes has become temporarily lost (Fig. 6). The final union appears to be nothing more than a contact. In some cells a tiny gap can be seen between the two chromosomes and is then often bridged by an achromatic connection—but it is difficult to decide whether such cases do not represent the initial step in the reductional separation of the *X* and *Y* in the second division (Fig. 7). In the late second anaphase the clubbed shape of both sex chromosomes is obliterated (Fig. 8).

It is natural to conclude that this interkinetic maneuver of the *X* and *Y* is in some way correlated with the properties of heterochromatin. But whether it is or not does not touch the question here at issue. Also it may be granted that the interplay of mitotic forces is to some degree acting in any grouping of chromosomes on the spindle (distance conjugation makes this almost certain). But by the same token the *Rhytidolomia* case furnishes strong evidence that the touch-and-go movement involves not only an attraction between the *X* and *Y*, but an attraction of a very specific type.

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[4] Darlington, C. D., *Jour. Genet.*, **39**, 101–136 (1939).
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THE CORRECTION OF X-RAY DIFFRACTION INTENSITIES FOR LORENTZ AND POLARIZATION FACTORS

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Communicated September 26, 1940

1. Introduction.—In an earlier contribution,[1] a method was suggested for recording x-ray diffraction results as F^2 values by direct photographic processes. This required the correction of the diffracted beams for Lorentz and polarization factors, and it was suggested that this could be accomplished with the aid of a rapidly rotating cam or shutter. A study of the theory of applying this correction has made it plain that, with the correct choice of variable, the form of the Lorentz factor is the same for all layers except for scale, and therefore a correction for it can always be made with the aid of a single instrumental cam regardless of the layer analyzed. On the other hand, the form of the polarization factor is different from level to level, and correction for it can best be made in another way. The results of this study can be applied not only in the recording of x-ray diffraction results as F^2 values, but they are of immediate use in allowing for Lorentz and polarization effects in any method involving a rotating crystal completely immersed in an x-ray beam.

The Lorentz factor for single crystals was first given by Darwin[2] for the case of "equatorial" x-ray reflections. Cox and Shaw[3] subsequently showed how the Lorentz factor could be extended to non-equatorial reflections recorded by the normal-beam method. More recently, Tunell[4] has given a similar treatment in the extension of the Lorentz factor to non-equatorial reflections recorded by the equi-inclination method. A discussion of the Lorentz factor for the general-inclination case has only recently received attention by Bouman and de Jong.[5] They used a comparatively cumbersome method of deduction and unfortunately their results do not have a form which is very convenient to apply.

2. General Characteristics of the Lorentz Factor.—When a crystal is rotated in a beam of monochromatic x-radiation, the various planes of the crystal are caused to pass through positions in which they satisfy the condition for reflection. In general, the various planes do not occupy such positions for equal lengths of time. The total amount of x-radiation in each reflection is proportional to this time opportunity to reflect. In the several rotating crystal methods, the Lorentz factor is essentially this time factor.

The Lorentz factor is proportional to the time of reflection permitted to each reflection, or inversely proportional to the velocity with which the plane passes through the condition of reflection. Using L to represent the

Lorentz factor and V the velocity of passing through the reflecting condition for the reflection hkl,

$$L_{hkl} \sim \frac{1}{V_{hkl}}. \tag{1}$$

The individual velocity of each reflection can be most conveniently investigated with the aid of the reciprocal lattice.

3. Reciprocal Lattice Derivation.—In figure 1 the important reciprocal lattice aspects of the problem are shown: A shows plan, C shows an elevation, while B shows an intermediate projection. The reciprocal lattice is rotating with an angular velocity, Ω. Reciprocal lattice point, P, on the nth level, has just reached the n layer reflecting circle and is in the process of reflecting. P is moving in the direction PW and has an absolute velocity of $\Omega\xi$. The velocity with which it passes through the sphere, however, is the component of this velocity on the normal to the sphere at the point P, namely, its component on PS. Calling η the angle between PS and PW, the speed with which P passes through the condition of reflection is therefore given by

$$V = \Omega\xi \cos \eta. \tag{2}$$

To evaluate cos η, pass a plane through the center of the sphere S and normal to the velocity direction, PW. It is evident that

$$\cos \eta = q. \tag{3}$$

Substituting this in (2) gives

$$V = \Omega\xi q. \tag{4}$$

Turning to figure 1-A, it will be observed that the last two terms of (4) represent twice the area of triangle O_nPS_n. This same area can be represented alternatively by R_0p, where R_0 is the radius of the zero-level reflecting circle. Substituting R_0p for ξq, (4) becomes,

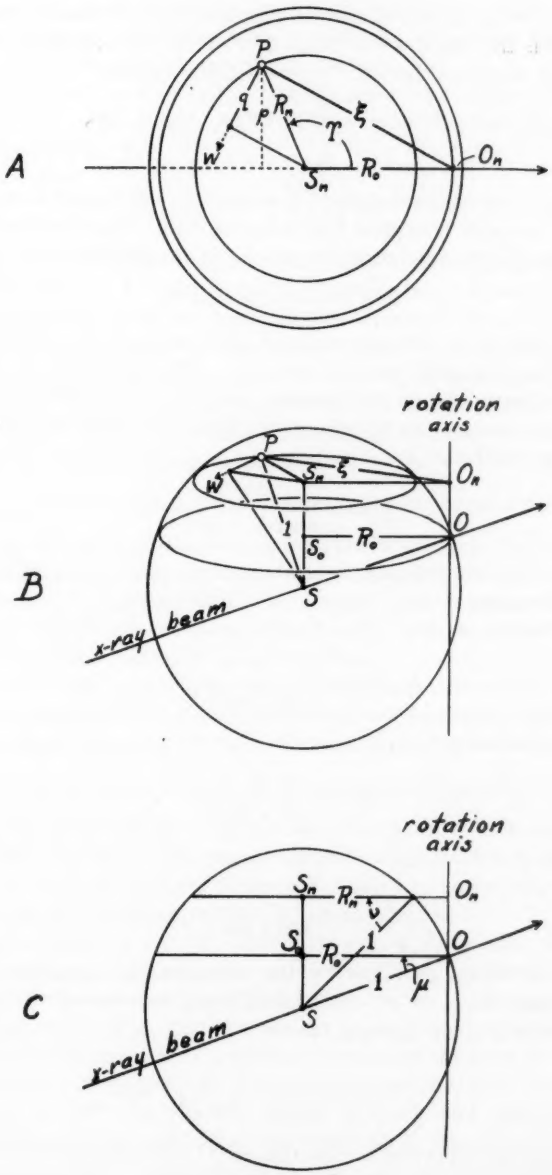
$$V = \Omega R_0 p. \tag{5}$$

This is in an inconvenient form. To transform it into something more useful, note that,

$$\sin \Upsilon = \frac{p}{R_n}, \tag{6}$$

where R_n is the radius of the reflecting circle of the level containing P. Substituting in (5) the value of p indicated by (6) yields the required velocity in quite fundamental terms:

$$V = \Omega R_0 R_n \sin \Upsilon. \tag{7}$$


A
P
q
p
R_n
ξ
T
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S_n
R_o
O_n
B
rotation axis
P
w
S_n
ξ
O_n
1
S_o
R_o
O
S
x-ray beam
C
rotation axis
S_n
R_n
O_n
S_o
R_o
1
O
1
μ
S
x-ray beam

FIGURE 1

Finally, the terms R_0 and R_n may be expressed as functions of μ and ν, the angles which the direct x-ray beam and diffracted cone make with the levels of the reciprocal lattice. Figure 1-*C* shows that,

$$R_0 = \cos \mu, \tag{8}$$

and

$$R_n = \cos \nu. \tag{9}$$

Since μ and ν are values required for setting the instrument to record the layer, it is desirable to express V in terms of them. Substituting from (8) and (9) into (7) gives the required velocity in the desired form:

$$V = \Omega \cos \mu \cos \nu \sin \Upsilon. \tag{10}$$

The Lorentz factor for any condition of recording is now given by substituting the value of V from (10) into (1). The result may be simplified for practical experimental purposes by noting that the velocity of crystal rotation, Ω, is common to all reflections; hence this term may be omitted from the proportion:

$$L \sim \frac{1}{\cos \mu \cos \nu \sin \Upsilon}. \tag{11}$$

This reduces to the following special cases for particular methods of recording reflections:

Normal beam, $\mu = 0$:

$$L \sim \frac{1}{\cos \nu \sin \Upsilon}. \tag{11A}$$

Equi-inclination, $\mu = -\nu$:

$$L \sim \frac{1}{\cos^2 \nu \sin \Upsilon}. \tag{11B}$$

Flat-cone, $\nu = 0$:

$$L \sim \frac{1}{\cos \mu \sin \Upsilon}. \tag{11C}$$

For equator reflections taken by the normal beam, equi-inclination or flat-cone methods, $\mu = \nu = 0$, and (11) can be reduced to the simple Darwinian form of the Lorentz factor:

$$L \sim \frac{1}{\sin \Upsilon} = \frac{1}{\sin 2\theta}. \tag{11D}$$

It is now most important to note that in (11) the pair of terms $\cos \mu \cos \nu$ is constant for any level, recorded by any method. These terms may be called the scale of the Lorentz factor for the level. An important property

of (11) is that, except for scale, the form of the Lorentz factor is exactly the same for every layer and for every method of recording, provided that the comparison is made for the correct variable, Υ.

4. Application to Weissenberg Photographs.—The vertical linear measure, x, of any Weissenberg photograph is directly proportional to the reflection azimuth, Υ:

$$\Upsilon = \left[\frac{360^\circ}{2\pi r_F}\right] x, \qquad (12)$$

where the term in brackets is an instrumental constant dependent on the camera radius, r_F. In order to find the Lorentz factor for any spot on any Weissenberg film, therefore, the only measurement necessary is the distance, x, of the spot from the center line. Since the instrumental constant in (12) is exactly 2 for the standard Weissenberg camera radius, the angle, Υ, can ordinarily be read directly with the aid of a millimeter scale. To find the Lorentz factor for the spot, it is merely necessary to take the cosecant of this angular reading and multiply it by the level scale in the form, $\frac{1}{\cos\mu\cos\nu}$, which is characteristic of the entire level and of the method of recording. Or if F^2 is required, as for Fourier analysis, the correction is $\sin\Upsilon$ multiplied by the scale $\cos\mu\cos\nu$. Either the Lorentz factor or the correction for it, disregarding the level scale, may be easily read for each spot on the film from a transparent chart (which is the same for all Weissenberg photographs) laid on or under the film.

5. Elimination of Computation of Level-Scale Correction.—Since the level-scale is constant for the entire level, it may be easily compensated for by deliberately arranging the time of exposure of each level so as to cancel it. The Lorentz factor varies from level to level by the scale, $\frac{1}{\cos\mu\cos\nu}$. To compensate for this it is only necessary to expose each layer, regardless of how recorded, for a time proportional to $\cos\mu\cos\nu$. When this precaution is observed, all level photographs taken by any method whatever have the same Lorentz correction for equal values of Υ.

6. Elimination of Lorentz Corrections during Recording.—In order to convert diffraction intensities into F^2 values, it is necessary to apply a correction for both Lorentz and polarization factors. This is usually done by computation. The computation can be entirely avoided as far as the Lorentz factor is concerned by taking advantage of the semi-invariant form of the factor. Neglecting, for the moment, the matter of scale, we note that at $\Upsilon = 90^\circ$ the Lorentz factor is unity and that it increases symmetrically above and below this value till it attains infinity at $\Upsilon = 0^\circ$ and $\Upsilon = 180^\circ$. A correction for this Lorentz enhancement can be applied during recording by suppressing the enhanced reflections by a

factor which is the reciprocal of the enhancing (Lorentz) factor. This can be accomplished by reducing the time of recording of the more intense reflection with the aid of a rapidly rotating cam. The proper characteristics of the cam are that it has an opening proportional to sin Υ. It is comparatively easy to arrange for such a cam in the method of de Jong and Bouman and also in the Sauter method. The writer has successfully applied it to de Jong and Bouman photographs and will provide further details of the method in another publication. On the other hand, it is difficult to arrange a rotating cam for the Weissenberg method, but it may be replaced here by a rapidly vibrating cylindrical shutter whose function is to differentially open and close the layer line screen slit. The design of such a shutter is complicated by the fact that the motion is reciprocating and hence is likely to be non-uniform.

The Lorentz correction may also be applied by absorbing reflections differentially as a function of Υ. The transmission of such an absorber should be proportional to sin Υ.

Regardless of what method is used to reduce reflections by the factor sin Υ, the Lorentz correction can be completely eliminated for any level and for any inclination scheme by simply arranging to have the exposure times for the levels proportional to cos μ cos ν, as discussed in section 5.

7. Note on the Elimination of Polarization Factor Correction.—The polarization factor can be eliminated by permitting transmission of reflections proportional to $\frac{2}{1 + \cos^2 2\theta}$. While this could be controlled with the aid of a cam, the function depends in an unfortunate way upon Υ, μ and ν, so that a different cam would have to be used for each level and a new set for each new crystal. It should be noted, however, that the correction factor $\frac{2}{1 + \cos^2 2\theta}$ has a range of only 1 to 2. It can therefore be easily applied by surrounding the crystal with a more or less spherical envelope of slightly absorbing material (such as a plastic) whose thickness variation has been designed to permit transmission of reflections by a factor $\frac{2}{1 + \cos^2 2\theta}$.

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FUCUS CORDATUS TURNER

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Communicated October 4, 1940

The *Fucus cordatus* of Turner (*Fuci*, **2**, 118, 119, pl. 116 (1809)) has been a much misunderstood plant and its various misinterpretations have been responsible for other misconceptions. The species was described from specimens collected by Mr. Menzies at "Banks' Isles" on the northwest coast of North America on the second voyage of Vancouver. "Banks' Isles," now generally known as Banks Island, lies on the east side of Hecate Strait in British Columbia and at about 50° lat. N. and 130° long. W. The total distribution of this species, according to one or another author, extends from the northeastern coasts of Asia, around through the type locality, along the west Pacific Coast of North America to (or almost to) the southern boundary of the United States, the coasts of Chile to the Antarctic continent and up along the eastern coast of South America to (and including) the Falkland Islands (cf. particularly Gain, "Deux. Exp. Antarct. française," 1908–1910, "La Flore Algologique," 54–58, 1912) and even to South Africa. Such a range even among the Red Algae is almost unparallelled, but from the point of view of moderate polymorphism did not, until very recently, seem beyond plausibility.

Fucus cordatus was referred by Bory (*Dict. Class.*, **9**, 16 (Feb., 1826)) to his proposed new genus *Iridea* as *Iridea cordata* and was the first of the binomials coined by him under this genus, although he mentioned the *Delesseria edulis* of Lamouroux first of all the species belonging to *Iridea*, but without coining a proper binomial for it. In Bory's time, fixing a type for a genus was not customary. Greville, however, in 1830 (*Alg. Britt.*, 136) definitely designates *Fucus edulis* Stackhouse as the type of the genus, as he had a perfect right to do. Later the name *Iridea* (or *Iridaea*) was more and more restricted to plants of the general type of *Fucus cordatus* and in 1847 J. G. Agardh (*Kongl. Vet. Akad.*, Handl., 82) in discussing the various types of South African plants still referred to *Iridea*, in his turn, formally designated *Fucus cordatus* Turner as the type of the genus, following the seeming usage of Bory in his latest account (*Voy. Coq., Bot.*, **2**, 103–114 (Feb., 1828)) and of Kuetzing, who (*Phycol. gen.*, 395 (1843)) places *Fucus cordatus* first in the enumeration and in his illustration (loc. cit., pl. 77, II, not truly of *Fucus cordatus* Turner) of the genus, although he includes *I. edulis* (Stackh.) Grev. and others which J. G. Agardh (loc. cit.) finally excluded from his conception of *Iridaea*.

When the late Professor Nathaniel Lyon Gardner and myself, after somewhat over thirty years of collecting and study, issued our preliminary

reports (*Proc. Nat. Acad. Sci.*, **22,** 469–473 (Aug., 1936), *Univ. Calif. Pub. Bot.*, **19,** No. 6 (May, 1937) and *Proc. Nat. Acad. Sci.*, **23,** 169–174 (May, 1937)), we felt that we had made progress in the way of distinguishing the various species of *Iridaea* from one another and also we found it desirable to assign a new name, *Iridophycus*, to the genus in order to avoid the confusion attending the earlier *Iridea* of Stackhouse (1816) as well as the confusion attending the designation of *I. edulis* (Stackh.) Grev. (1830) and *I. cordata* (Turn.) Bory by J. G. Agardh (1847) as types of the genus. Also, in order to avoid confusion, we designated *I. capensis* J. Ag. as the type under the new generic name, since type specimens of this species were in existence and no type specimen of *Fucus cordatus* was known.

The Hooker Herbarium at Kew, after exhaustive searching both by myself and by the staff at Kew, failed to reveal any specimen which could be regarded as the type of *Fucus cordatus* Turner. The collections at the British Museum of Natural History likewise were searched to no avail and Gardner and I seemed justified in assuming that no type had been saved since Dawson Turner, himself, saved no types, according to Sir Joseph Dalton Hooker, his grandson, who states in a letter preserved at Kew: "My grandfather was not the sort of man to collect and preserve plants" (*Proc. Linn. Soc. London*, 150th session, pt. 1, 22 (Dec, 1937)). We may presume that such of Dawson Turner's Algae as are to be found in the "Hooker Herbarium" were preserved through the meticulous care of his son-in-law, Sir William Jackson Hooker, who is responsible for so many of the excellent drawings of Turner's "Fuci."

While there seemed to be no doubt from our previous studies that *Fucus cordatus* Turn. was not among the *Iridophycus* species known from the Southern Hemisphere, the question as to its exact characters became acute as Gardner and myself attempted to unravel the complex of the species of the coasts of western North America and northeastern Asia. Following out a suggestion of one of our fellow botanists (Dr. Ivan M. Johnston of the Gray Herbarium) that some of the plants collected by Archibald Menzies were to be found at the Royal Botanic Gardens at Edinburgh, a letter was addressed to the Regius Keeper, Sir William Wright Smith, with the result that a sheet of two specimens of "*Fucus cordatus*" collected by "A. M., Banks' Isles, N. W. Coasts of America" were found in the "Menzies Herbarium" and generously forwarded to us for study. The lower specimen answers so very closely to Turner's figure of the plant (*Fuci*, **2,** pl. 118) that it seems possible that it may be the very specimen from which the drawing was made. It is, moreover, just one-half the diameter of the figure of Turner, who says, however, that it "contracts to little more than half its original size" in drying. It seems, therefore, since in all its characters it corresponds to Turner's description, that here we have the type material, or if any doubt remains, at least duplicate type

material, and it is possible to study the cystocarpic type in all essential detail.

Fucus cordatus Turner, both as described by Turner and as represented by the Menzies specimens in Herb. Edinb. is a plant of the size of the series of larger species of *Iridophycus* (cf. Setchell and Gardner, *Proc. Nat. Acad. Sci.*, **23**, 174 (Mar., 1937)). Turner describes it as about a foot long and 6 inches wide, which corresponds to the larger of the two specimens in the Menzies Herbarium, after allowing about half diameter for shrinkage during drying. The fronds, according to Turner, arise from "a largish, expanded, callous disk" and "the fronds are numerous from the same base." There is a distinct stipe, about 1.2 cm. long ("6 lines," according to Turner) which is flattened, narrow below and expanding above into a narrow cuneate apophysis, distinctly set off from the stipe below and the broad flaring base of the lamina (or blade) above and about 1 cm. long and 0.6–1 cm. broad. The color of the whole plant from basal disc to apex is described by Turner as "livid-brown, not without a tinge of purple, transparent, darker when dry, and then looking almost black unless held up to the light," with "substance" described as between "coriaceous and fleshy, thick and full of moisture."

The blade is broadly ovate-oblong, with broad, shallow, cordate base, more or less unsymmetrical, narrowing in the upper third to a broad and obtuse point. The margins are devoid of projections or lobes, "quite entire" is Turner's description, but are broadly undulate, with the very small cystocarps of extraordinarily uniform size, scattered over the entire blade with the exception of a crescentic zone just above the apophysis. Turner says: "in the lower and narrowed part of the frond the margins are remarkably elevated and incurved and this part consequently looks not only like a stem, but a channeled one." This shows also in the Menzies specimens and indicates that the species has a very definite apophysis set off from both the blade above and the short stipe below.

The cystocarps are described by Turner (loc. cit., p. 118) as being "spherical tubercles, smaller than a pin's head, of a tawny color, scattered without order all over the frond, and immersed in the middle of its substance, containing a number of minute, oblong, red seeds, together with pellucid jointed fibers." While the general morphology of *Fucus cordatus* is almost sufficient to stamp it as a very well defined species among those of *Iridophycus*, the uniformity of the cystocarps together with their comparatively small size, add to the other indications that it is most distinct and, as will be indicated later, evidently restricted to a small region in northwestern North America.

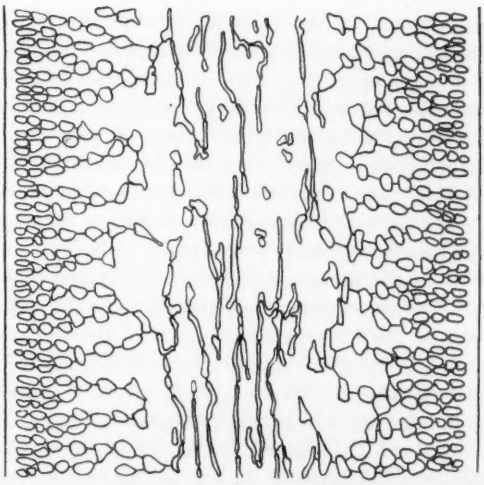
Transverse sections of the plant of Menzies are from about 186 to 245 μ thick in the sterile portions (when only moderately swollen up, to about what seems normal when fresh (i.e., cut by freezing method in 70 parts

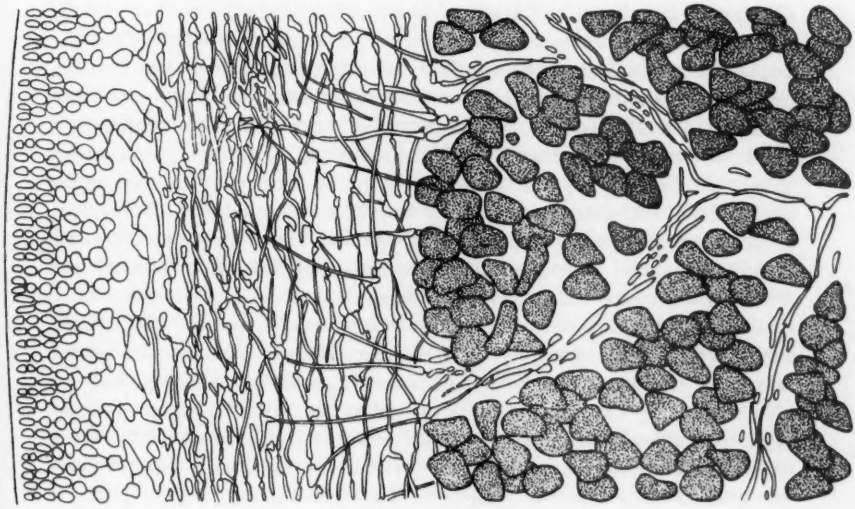
of glacial acetic acid and 30 parts of water, mounted in "Karo") while the cystocarpic portions are up to 375 μ thick or even somewhat thicker. That portion of the blade (see figure 1) which is not cystocarp-bearing shows a medullary layer occupying about three-sevenths of the diameter of the section and consisting in very thin section of about 4 more or less parallel slender and loosely placed filaments, about 3–4 μ diam. The segments of the hyphae are commonly elongated (up to 28 μ long). In thin sections cut about 15 μ thick and treated so as to prevent undue swelling, the filaments of the medulla show practically parallel (as in figure 1), but in thick sections (or crushed fragments) the hyphal segments form an irregular but close reticulum often with elongated meshes and with many free ends, enough to disturb the regularity of the reticulum. In thicker or more swollen sections the irregular network is definitely apparent but never approaches the regular reticulum of the *Iridaea cordata* of Kuetzing (*Phyc. Gen.*, 395, pl. 77, II (1843), *Tab. Phyc.*, 17, pl. 6, fig. *b* (1867)) from Valparaiso, which is obviously not Turner's species but *Iridophycus Boryanum* Setch. et Gard. (*Proc. Nat. Acad. Sci.*, **22,** 470 (1936)).

The cortex of the sterile portions of the frond is indistinctly double (as in most species of the genus), divided into an outer and an inner. The outer cortex consists of closely packed anticlinal rows (or anticlinal groups) of cells, 6–8 in number (vertical to the surface). These cells form a perfectly distinct layer of about 6 cells from the surface, the lower larger and rounded or slightly flattened, the upper 3 (or 4) slightly elongated vertical to the surface. The outermost, commonly single, may be globular to cartridge-shaped. Occasionally the outermost cells arise by twos from the cell below and are decidedly elongated as indicated by Kylin (*Lund Univ. Årsskr.*, n. f., Avd. 2, **24,** 47, fig. 25 (1928)).

The lower portion of the outer cortex is not, in cystocarpic and antheridial plants, sharply delimited, but between it and the medulla there is a tissue, indefined on outer and inner sides, of several layers of distant, horizontally elongated cells, short to fairly long in horizontal diameter (parallel to the surface of the frond) which exists in all species of *Iridophycus* and which, because of its augmentation in the tetrasporic plants of the genus, may be distinguished as an inner cortex. This indefined inner cortex equals approximately the thickness of the outer cortex. The comparatively slight development in the inner cortex in sterile, antheridial and cystocarpic individuals of species of *Iridophycus* as compared with tetrasporic individuals makes for the extreme increase in thickness found in the last and is to be emphasized here in the case of the type of *Fucus cordatus* since search for the proper tetrasporic type is utterly complicated as to thickness and complexity of structure.

The cystocarps in Turner's plant (cf. figure 2) measure up to about 500 μ





FIGURES 1 AND 2

Iridophycus cordatum (Turn.) S. et G.

Fig. 1 (*above*). Portion of a transverse section of a sterile portion of the blade of the "type" specimen. × 400 diam.

Fig. 2 (*below*). Portion of a similar section showing the upper segment of a cystocarp. × 400.

(*Drawn by Roy W. Donley under direction of the author*)

in horizontal diameter and about 315 μ in vertical diameter. The spores are in definite groups, separated by slender "nutritive" filaments, and the whole cystocarp is enclosed within a distinct and fairly thick (60–65 μ) layer of concentrically disposed filaments, the "*pericarpium proprium*," from whose filaments radiating "nutritive" filaments pass into the groups of spores (from gonimoblastic lobes) and join the nutritive filaments surrounding each of them (see figure 2). The *pericarpium proprium* arises from the inner cortex. The cystocarps form rather to one surface or the other of the frond, enlarging to become almost central at maturity and causing the surfaces of the frond to bulge out slightly and almost equally on both surfaces. The large, densely filled auxiliary cells ("Tragzellen") are shown in many of the immature cystocarps of the sections from the type material and in very characteristic fashion (cf. Kylin, "Anat. d. Rhodophyc.," figure 210A (1937)).

The cystocarpic type of *Iridophycus cordatum* (Turner) Setchell et Gardner (*Proc. Nat. Acad. Sci.*, **23**, 170 (Mar., 1937)) seems clearly established from the Menzies specimens in the Herbarium of the Royal Botanic Garden of Edinburgh. Its salient points are the distinct stipe, equally distinct broadly cuneate apophysis, the flaring, more or less shallow cordate base of the ovate blade with smooth but undulate margins, and the regularly and thickly scattered small cystocarps of uniform and small diameter. Added to this, the blade is only moderately thick and of a brownish purple color, becoming opaque and blackish on drying. Examining sections of dried material (cut by the freezing method in 70 parts of glacial acetic acid to 30 of water and mounted in "Karo," so as to prevent undue swelling), the medullary layer shows 3–4 (?) moderately slender filaments with usually a parallel course although anastomosing into an elongated and irregular reticulum with many free endings as seen in the thicker sections or on crushing. If only a little more water is added the sections enlarge very considerably and even become disorganized, as is the case with other species of the genus and even with most Gigartineae. The cortices are also reasonably distinctive, consisting of a few layers of larger, widely separated cells of the inner cortex and an outer cortex of about 6 cells in each anticlinal row. The cystocarpic portions of the fronds bulge slightly on both surfaces and the flattened, bluntly lenticular cystocarp has a distinct *pericarpium proprium*.

Studies of various plants, particularly those collected from the general Puget Sound region and northward, for occurrences of *Fucus cordatus* have been made with the following results: (1) no specimens from south of Cape Flattery; (2) a number of specimens from the west coast of Whidbey Island, Wash., collected by N. L. Gardner, which agree within fair limits; (3) some young specimens from Table and Calvert Islands, B. C., collected by T. T. and E. E. McCabe, which are in the general distribu-

tional area of Banks Island, B. C.; and (4) two specimens collected by John Macoun on Vancouver Island, B. C., without special data, but judging from Macoun's correspondence, probably in the neighborhood of Ucluelet on the west coast area about May, 1909. None of these various specimens is mature cystocarpic, but several are young or mature tetrasporic, while some may possibly be young antheridial. The lack of additional mature cystocarpic specimens (the type specimen being cystocarpic) for exact comparison and since tetrasporic plants of species of *Iridophycus* differ from cystocarpic especially in details of thickness and difference of histologic details due to the usually prominent tetrasporogenic layer, make the lack of any additional specimens from the exact type locality unfortunate.

Since both cystocarpic and tetrasporic plants are necessary to understand a species, it has seemed reasonable to assume that the tetrasporic generation is represented by such plants as were included by N. L. Gardner under his No. 58, cast ashore on the western shores of Whidbey Island, Wash. (Herb. Univ. Calif. Nos. 464012, 547647, 547648). Another mature tetrasporic plant is No. 72 of John Macoun, probably collected at or near Ucluelet, on the west or ocean side of Vancouver Island in 1909. All of these plants agree in general habit with the type specimen (cystocarpic) but differ in minor details of gross and naturally, of microscopic anatomy. They vary up to about twice the length of the type and in one at least the blade is very broad. The blade varies from a moderate flare above the apophysis to shallow cordate, while the tip is shallowly to fairly deeply and broadly lobed into two to three divisions. None is cleft to (or through) the apophysis. The color of the mature fruited plants is deep claret to almost black in the dried specimens.

The stipe is comparatively long (1.2–1.5 cm.) and about 1.5 mm. thick, cylindrical at the very base, compressed above. The apophysis is distinct, broad cuneate, 1–2 cm. long and about 2 cm. broad above. The blade flares suddenly and with a shallow cordate to reniform base. A similar habit, to a certain degree at least, is to be found in *I. coriaceum* S. et G., *I. splendens* S. et G., *I. lineare* S. et G., *I. whidbeyanum* S. et G., *I. fulgens* S. et G. and *I. sinicola* S. et G., but the coarse, crowded, short and irregular segments of the medulla of the blades of these species easily distinguish them, as well as minor differences in stipe, apophysis and blade. The nearest approach to *I. cordatum* (Turn.) S. et G. is *I. flaccidum* S. et G., but there are differences in color, stipe, apophysis, thickness and medullary structure sufficient to separate characteristic plants of this species although, at times, there may be difficulty in placing young or sterile specimens.

Tetrasori in the plants selected as typical (N. L. Gardner No. 58, as indicated above) are very thickly and regularly scattered over the whole

blade (at maturity) except that portion of it immediately above the apophysis. In more translucent dried specimens they appear as fairly uniform, small, dark, circular to elongated-oblong patches, on both surfaces, each of which is slightly elevated above such sori as occur directly beneath it. The mature sori are separated from one another by intervals less than that of their own diameters. In cross-section the tetrasporic plants show the same type and thickness of medulla as found in the cystocarp type plant, but differ, as do all the tetrasporic plants of the genus *Iridophycus*, in details of the outer and inner cortices. The outer cortex of the tetrasporic plant differs from that of the cystocarpic plant in having the terminal cells of the anticlinal rows more regularly in pairs and somewhat to very considerably elongated, a matter of only occasional occurrence in cystocarpic plants, and giving their cortices a decidedly different aspect. The most modifying difference between cystocarpic and tetrasporic plants, however, is in the voluminous development of the tissue between the medulla (proper) and the outer cortex (proper), since as in all the species of *Iridophycus*, this "inner cortex," not very distinctly set off from the "outer cortex" in the sterile, antheridial and cystocarpic plants, increases in thickness, and through the multiplication of cells forming a complex of elongated and connecting filaments between the medullary segments on the one side and the anticlinal rows of the cortex on the other, a definite sporogenic tissue is formed up to about 100 μ to 150 μ in thickness (on each side). In this tetrasporogenic plexus, the intercalary cells enlarge, multiply, pull away from one another and form finally the tetrasporangial mother cells, from which by division at right angles the tetraspores are formed. The tetrasporangia are thus intercalary or insterstitial as to origin, but are "accessory" in the sense that they occur as a multiplication of cells of the inner cortex (not of the outer cortex or of the medulla) into a distinct tetrasporogenic tissue.

As may be understood from the statement of the intercalation of a distinct and relatively thick tetrasporogenic tissue on each side of the medulla, the thickness of a fertile (or even immature) tetrasporic plant will greatly exceed that of the sterile portions of the cystocarpic plant. While the sterile portions of a tetrasporic plant may be little over 250 μ as in the cystocarpic plant, the thickness of the greater part of a tetrasporic plant (according to stages toward maturity) may increase up to 685 μ or less. In such a section the medulla accounts for about 63 μ of the thickness and each of the outer cortices for about the same, while each tetrasporogenic tissue, in sterile portions of the blade only about 20–25 μ as an inner cortex, has become 150–200 μ thick, according to its maturity. The tetrasporogenic layers are not absolutely continuous, but slight breaks may occur in them.

The sori are distinct, flattened spheroid, usually about 60–65 μ in

vertical diameter at maturity and vary in their horizontal diameter from about the same up to 625 μ and (exceptionally) even up to 1000 μ. They impinge upon the true medulla inwardly and upon the outer cortex outwardly, and cause bulges, above each, of the outer cortex. The tetrasporangia are somewhat variable in size, elongated ellipsoid, about 28 $\times$ 14 μ divided cruciately into the 4 tetraspores.

Summary.—It has seemed best to consider the *Fucus cordatus* Turn. in sufficient detail to make clear its nature, since it has been regarded as:

1. The morphological and even the taxonomic type of the genus *Iridaea* of Bory;
2. The species of widest distribution through Pacific North and South America and even South Africa;
3. A species whose type specimen had been lost, until the discovery of the Menzies specimen in Herb. Edinburgh.

With the discovery of that type and the realization that there were somewhere about 30 species of *Iridophycus* segregated in at least 4 distinct sections (or subgenera?), the necessity for some exact knowledge of the nature and distribution of *Fucus cordatus* Turn. became imperative. There results from this study, not only of the cystocarpic type specimen but also of the tetrasporic plants seemingly undoubtedly to be referred to it, the following:

1. That it has a distinct, relatively long stipe;
2. That its apophysis is broad cuneiform, distinctly set off from both the stipe below and the blade above;
3. That the blade is rather broadly cordate, with some variation as to flare at the base;
4. That the tip is broadly and obtusely pointed, either entire or 2–3-lobed;
5. That it belongs to the group of larger species of *Iridophycus;*
6. That its medullary tissue is of slender filaments, anastomosing into an irregular reticulum, and to be distinguished by this from species of similar aspect whose medullary filaments are coarse, short articulate and not forming apparent reticula;
7. That the tetrasporogenic inner cortex is highly developed in the tetrasporic plant, as in other species of *Iridophycus* where this tissue is continuous;
8. That both the mature and the immature cystocarps are typical of *Iridophycus;*
9. That its distribution is most probably confined to the shores of northwestern North America, not having been found south of the Straits of Juan de Fuca, and certainly bearing no relation to the South American or South African species earlier considered identical with it.

EVERY TWO EQUAL ORDER SUBGROUPS HAVING ONLY IDENTITY IN COMMON

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Communicated October 10, 1940

The term subgroup will be used in the present article for a proper subgroup, that is, for a subgroup which is neither the identity nor the entire group. Suppose that a given abelian group G has a set of subgroups of the common order p^k, p being a prime number, such that every pair of these subgroups has only the identity in common and that all the operators of G appear in these subgroups. The order of G is then of the form p^m. To prove that m is divisible by k it may first be noted that such a G contains an invariant subgroup of order p^k and that if this subgroup is extended by another such subgroup contained in G there results an invariant subgroup of order p^{2k} when $m > 2k$. Similarly there appears in G an invariant subgroup of order p^{3k} whenever $m > 3k$, etc. Hence there results the theorem that *whenever a group contains a set of subgroups of order p^k, p being a prime number, such that all the operators of the group appear in these subgroups and every two of them have only the identity in common, then its order is p^m and $m = nk$, where n and k are positive integers.*

The simplest case presents itself when all the operators of G besides the identity are of order p irrespective of whether G is abelian or non-abelian. The number of the subgroups of order p in such a G is $(p^m - 1)(p - 1)$. We proceed to prove that whenever G is an abelian group of order p^m and of type 1^m it is always possible to find in G a set of subgroups of the common order p^k such that every operator of G besides the identity appears in one and only one of these subgroups whenever $m = kn$ where k and n are positive integers. It was proved above that this is a necessary condition. To prove that it is also sufficient we may represent G as the direct product of n regular permutation groups of order p^k and of type 1^k and note the number of the subgroups of order p^k contained in this direct product which satisfy the condition that every pair of these subgroups has only the identity in common. It is clear that there are n such regular subgroups and that these involve all the permutations of degree p^k contained in G.

The permutations of degree $2p^k$, which appears in such a G, are found in the simple isomorphisms between sets of two such regular groups. In the particular case when $n = 2$ we thus obtain $1 + p^k$ subgroups which have only the identity in common since the group of isomorphisms of the abelian group of order p^k and of type 1^k involves a cyclic group of order $p^k - 1$. The number of the subgroups of degree np^k and of order p^k in

which all the permutations besides the identity are of degree np^k is $(p^k - 1)^{n-1}$ since these may be obtained by establishing simple isomorphisms between the given n regular group of order p^k. By establishing simple isomorphisms between the n sets of $n - 1$ simply isomorphic regular groups of order p^k there result $n(p^k - 1)^{n-2}$ groups which have only the identity in common with each other and with the given $(p^k - 1)^{n-1}$ groups of degree np^k, etc. Hence the total number of the subgroups which have separately only the identity in common and involve all the permutations of G is

$$p^{(n-1)k} + p^{(n-2)k} + \ldots + 1 = (p^{nk} - 1)/(p^k - 1).$$

It is easy to prove that when G is an abelian group of order p^m which involves operators whose orders exceed p then it is impossible to find in G a set of subgroups of order p^k, $k > 1$, such that every pair of these subgroups has only the identity in common and that these subgroups involve all the operators of G. This follows from the fact that such a group contains operators of order p^2 and that all the products of one of its operators of order p^2 into an operator of order p generate the same subgroup of order p. If a subgroup which involves a cyclic subgroup of order p^2 has only the identity in common with every other subgroup of the set to which it belongs it therefore involves all the operators obtained by multiplying this cyclic subgroup of order p^2 by all the operators of order p contained in the group. Hence such a subgroup involves all the operators of order p contained in the group. As every other subgroup of the set involves operators of order p and has only the identity in common with this subgroup we have arrived at a contradiction by assuming that in an abelian group of order p^m which involves operators of order p^2 it is possible to find a set of subgroups involving all the operators of the group and satisfying the condition that every pair of subgroups has only the identity in common.

If the order of an abelian group G is divisible by at least two distinct prime numbers it is not possible to find therein a set of subgroups of the same order which involve all the operators of G but involve no two subgroups which have more than the identity in common. If such a set would exist in G each of the subgroups of the set would involve a cyclic subgroup whose order would be divisible by all the prime numbers which divide the order of G and therefore by the product of all these prime numbers. If p is such a prime number no other subgroup of the set could involve an operator of order p. Hence we have arrived at a contradiction by assuming that the order of G is divisible by two distinct prime numbers and it has been proved that *the only abelian groups which satisfy the condition that they involve a set of subgroups of the same composite order which include all the operators of the group but include no two subgroups which have more*

than the identity in common are the abelian groups of order p^m and of type 1^m, p being a prime number.

We proceed to prove that every Sylow subgroup of each one of the subgroups of equal order which have no common operator besides the identity and include all the operators of G is abelian and of type 1^k. It may first be noted that if such a subgroup were non-abelian it would be transformed into itself by an operator in the corresponding Sylow subgroup of G which does not appear in this subgroup in view of the theorem that in a prime power group every proper subgroup is transformed into itself by an operator of the group which does not appear in this subgroup. As this transforming operator would also appear in one of the set of subgroups which involve all the operators of G but contain no common operator besides the identity it follows that the latter subgroup would have an operator besides the identity in common with the former which is contrary to the hypothesis. It therefore results that *all the operators of G which are powers of the same prime number and appear in the subgroups involving no common operator besides the identity but involving all the operators of G are of prime order.*

Suppose that G is a non-abelian group of order p^m, $p > 2$, which involves no operator of order p^2 but contains an abelian subgroup of index p and a commutator subgroup of order p. The central of G is then of order p^{m-2}. To prove that not all the operators of this G can appear in subgroups of order p^2 such that every two of these subgroups have only the identity in common it may be noted that if a subgroup of order p^2 is in a subgroup of index p under G which includes the central but is not itself in this central then it has just p operators in common with this central. The number of such possible subgroups is therefore $(p^{m-2} - 1)/(p - 1)$ and the total number of the operators of G which appear in these subgroups constitute a subgroup of index p under G. It has therefore been proved that in a non-abelian group of order p^m which involves no operator of order p^2 but has an abelian subgroup of index p and a commutator subgroup of order p it is not possible to distribute all the operators into subgroups of order p^2 such that every pair of these subgroups has only the identity in common.

It is not difficult to extend this result so as to apply to abelian subgroups of any larger order contained in such a G. If the order of such an abelian subgroup would be p^k it results that m would be nk for the same reason as when G was assumed to be abelian. The operators of such a subgroup which would appear in the central of G would constitute a group of order p^{k-1} just as in the case when $k = 2$. Hence all the operators of the central would be exhausted by the subgroups found in a subgroup of index p under G. It would therefore not be possible to construct the remaining subgroups of order p^k so that all the operators of G would appear in these

subgroups and every two of these subgroups would leave only the identity in common.

It can now be easily proved that the Sylow subgroups of G are abelian and of type 1^k. It was proved above that all the operators of such a group, besides the identity, are of prime order. If the group were non-abelian its operators would appear in sets of conjugate subgroups having only the identity in common since G could not be a Hamiltonian group. These subgroups would be transformed according to a transitive permutation group under this Sylow subgroup. The subgroups which correspond to the letters of this permutation group could not all be transformed into different such subgroups by any one of these subgroups. They would therefore be thus transformed by an operator of this Sylow subgroup which is found in the other such subgroup since every transitive permutation group involves permutations which permute all the letters of the group.

This operator would transform this set of conjugate subgroups among themselves and hence it would transform the non-invariant operators of the set into operators of the set. The subgroup of the given common order of distinct subgroups which involve all the operators of the given Sylow subgroup to which the operator in question belongs would therefore be transformed into itself multiplied by an operator found in the given conjugate set of subgroups. As this is impossible it results that the Sylow subgroup in question is abelian and hence there results the following theorem: *If a non-abelian group has the property that all of its operators appear in a set of subgroups of the same order such that every two of these subgroups have only the identity in common then the Sylow subgroups of this group are abelian and of type* 1^k.

GENERALIZED HOMOLOGY GROUPS

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Communicated October 8, 1940

We propose the following generalization of homology groups:

In a topological space M we define an unoriented *singular* simplex in the usual manner as a continuous map of a euclidean simplex onto M (Seifert-Threlfall, p. 92). As the coefficient group of the chains corresponding to the singular simplexes, we choose the group of all integers modulo a prime integer p.

We define the *boundary* of a singular simplex A^i $(i > 0)$ to be the chain consisting of the $(i - 1)$-dimensional (singular) faces of A^i where each

face is taken with the coefficient *plus one*. This definition is equivalent to the usual one only for the case $p = 2$. The case $p = 3$ is discussed below. The boundary $B(K^i)$ of a chain K^i is the sum, modulo 3, of the boundaries of the constituent simplexes of K^i.

It is easy to verify that (for $p = 3$) the third boundary of every chain vanishes (and not the second boundary as in the case $p = 2$), i.e.,

$$B\{B[B(K^i)]\} = 0. \tag{1}$$

Hence, two kinds of "cycles" exist, namely the cycles C^i of type α whose first boundaries vanish and the cycles C^i of type β whose second boundaries vanish. Of course, a cycle of type α is also of type β (a 1-chain is, by definition, a cycle of type β, a 0-chain, a cycle of type α). According to equation (1), the chain $B[B(K)^{i+2}]$ is an i-cycle of type α and $B(K^{i+1})$ an i-cycle of type β.

For the case $p = 3$ two generalizations of the homology groups present themselves for study (and for the case of an arbitrary integer p, $p - 1$ generalizations). The first homology group of dimension i is the group of all i-cycles of type α modulo the subgroup formed by the cycles of type α that are second boundaries of $(i + 2)$-chains: namely, that of the form $B[B(K^{i+2})]$. The second homology group is the group of all i-cycles of type β modulo the subgroup formed by the cycles of type β that are first boundaries of $(i + 1)$-chains: $B(K^{i+1})$. Obviously the groups here defined are topological invariants.

We have proved that if the space M is a complex of simplexes the homology groups of M can be derived by using only the chains determined by the simplexes of the complex (provided we always include "degenerate" simplexes with repeated vertices). For the case $p = 3$, these groups remain unchanged if we use only simplexes with vertices of multiplicity no greater than 2. In other words, a simplex with a vertex of multiplicity of 3 or more plays the same rôle (for the case $p = 3$) as a degenerate simplex in the classical theory. That is to say, all simplexes of this sort may be identified with the chain 0 without altering the homology groups.

This is the main result so far obtained. A detailed exposition will shortly be given in another journal. The proofs, as in the case of the ordinary homology groups, are based on the fundamental idea of a simplicial approximation.

FUNCTIONS WHOSE EVEN DERIVATIVES HAVE A PRESCRIBED SIGN

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Communicated October 17, 1940

R. P. Boas has suggested to the author that one might hope to use Lidstone series to prove the analyticity of a function whose even derivatives do not change sign on a given interval. The original idea was to prove that if these even derivatives are all positive then the function is analytic on the interval in question. The method turned out to be of no use for this particular problem, but Boas[1] gave a proof of the result independently by another method. However, his suggestion led the author to prove the wholly unexpected result that if

$$(-1)^k f^{(2k)}(x) \geqq 0 \qquad (k = 0, 1, 2, \ldots) \tag{1}$$

on any interval, however small, then $f(x)$ is entire. The functions $\sin x$ and $\cos x$ are illustrative examples.

In this note we give a complete proof of the above result, using a minimum of the theory of Lidstone series. In a later paper we shall go into the relation of the present result to the problem of the representation of functions by such series.

The Lidstone series for a given function $f(x)$ is defined as

$$\sum_{n=0}^{\infty} f(1)\Lambda_n(x) + f(0)\Lambda_n(1 - x),$$

where

$$\Lambda_0(x) = x$$

$$\Lambda_n''(x) = \Lambda_{n-1}(x)$$

$$\Lambda_n(0) = \Lambda_n(1) = 0 \qquad (n = 1, 2, \ldots).$$

These conditions determine the polynomials $\Lambda_n(x)$ completely.

Define the functions $G_n(x, t)$ as follows:

$$G_1(x, t) = (t - 1)x \ (t > x)$$

$$= (x - 1)t \ (t < x)$$

$$G_n(x, t) = \int_0^1 G_1(x, u)G_{n-1}(u, t)du \qquad (n = 2, 3, \ldots).$$

It is easily seen that

$$\Lambda_n(x) = \int_0 G_n(x, t)t\,dt.$$

With these preliminaries we now prove our result by a series of theorems.

THEOREM 1. *If* $f(x) \in C^{2k}$ *on* $(0 \leqq x \leqq 1)$, *then*

$$f(x) = \sum_{n=0}^{k-1} f^{(2n)}(1)\Lambda_n(x) + f^{(2n)}(0)\Lambda_n(1-x) + R_k(x) \qquad (2)$$

$$R_k(x) = \int_0^1 G_k(x, t) f^{(2k)}(t)dt. \qquad (3)$$

This is proved by integrating the integral (3) by parts.

Now observe that for any positive integer k

$$(-1)^k G_k(x, t) \geqq 0 \qquad (0 \leqq x \leqq 1, 0 \leqq t \leqq 1),$$

so that if (1) holds on $(0 \leqq x \leqq 1)$, then all the terms of (2) are non-negative. This is the essential fact which enables us to obtain bounds on the successive derivatives of $f(x)$.

THEOREM 2. *For any positive integer* k

$$(-1)^k \Lambda_k(x) \geqq \frac{x^{k+1}(1-x)^{k+1}}{(k+2)!} \qquad (0 \leqq x \leqq 1). \qquad (4)$$

Observe first that

$$(-1)^k \int_0^1 [G_1(x, t)]^k \, dt = \frac{x^k(1-x)^k}{k+1}$$

and that

$$-G_1(x, t) \leqq t(1-t) \leqq t \qquad (0 \leqq x \leqq 1, 0 \leqq t \leqq 1).$$

Hence

$$-\Lambda_1(x) = -\int_0^1 G_1(x, t)t \, dt \geqq \int_0^1 [G_1(x, t)]^2 dt = \frac{x^2(1-x)^2}{3} \geqq \frac{x^2(1-x)^2}{3!}.$$

That is, (4) is true for $k = 1$. Now assume it true for $k = n - 1$. Then

$$\begin{aligned}
(-1)^n \Lambda_n(x) &= (-1)^n \int_0^1 G_n(x, t)t \, dt \\
&= (-1)^n \int_0^1 G_1(x, u)du \int_0^1 G_{n-1}(u, t)t \, dt \\
&= (-1)^n \int_0^1 G_1(x, u)\Lambda_{n-1}(u)du \geqq -\int_0^1 G_1(x, u) \frac{u^n(1-u)^n}{(n+1)!} du \\
&\geqq \frac{(-1)^{n+1}}{(n+1)!} \int_0^1 [G_1(x, u)]^{n+1} du = \frac{x^{n+1}(1-x)^{n+1}}{(n+2)!}.
\end{aligned}$$

Thus the proof is completed by induction.

THEOREM 3. *If* (1) *holds on* $(0 \leqq x \leqq 1)$, *then*

$$f^{(2k)}(1) = O[4^{k+1}(k+2)!] \qquad (k \to \infty). \qquad (5)$$

For, from (1) and (2) we have

$$0 \leqq f^{(2k)}(1)\Lambda_k(x) \leqq f(x) \qquad (0 \leqq x \leqq 1).$$

Setting $x = 1/2$ and making use of (4) we have (5).

THEOREM 4. *If* (1) *holds on* $(0 \leqq x \leqq 1)$, *then there is a constant* M *such that*

$$(-1)^k f^{(2k)}(x) \leqq M4^{k+1}(k+2)!/x^{2k} \qquad (0 < x \leqq 1). \tag{6}$$

For, if $f(x)$ satisfies (1) on an interval $(0 \leqq x \leqq b)$, then $F(x) = f(bx)$ does so also on the interval $(0 \leqq x \leqq 1)$. Applying Theorem 3 to $F(x)$ we have

$$F^{(2k)}(1) = b^{2k} f^{(2k)}(b) = O(4^{k+1}(k+2)!). \tag{7}$$

Replacing b by x in (7) gives (6).

THEOREM 5. *If* (1) *holds on* $(0 \leqq x \leqq 1)$, *then* $f(x)$ *is entire.*

For, from (6) we see that

$$f^{(2k)}(x) = O(4^{k+1}(k+2)!) \qquad (k \to \infty) \tag{8}$$

uniformly in $(a \leqq x \leqq 1)$, where a is any positive number less than unity. By a familiar relation[2] between the bounds of the even derivatives and the bounds of the odd derivatives we have

$$f^{(2k+1)}(x) = O(4^{k+2}(k+3)!) \qquad (k \to \infty) \tag{9}$$

uniformly in $(a \leqq x \leqq 1)$. Relations (8) and (9) are sufficient to show that the Taylor expansion of $f(x)$ converges to $f(x)$ for all x.

This result has the following interesting consequence.

THEOREM 6. *If*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n,$$

the coefficients a_n *being real and such that*

$$(-1)^n a_{2n} > 0 \qquad (n = 0, 1, 2, \ldots)$$

$$\overline{\lim_{n \to \infty}} \sqrt[n]{|a_n|} > 0,$$

then in any interval $(0 \leqq x \leqq \delta)$, *however small, some even derivative of* $f(x)$ *changes sign.*

[1] This proof will appear in an early issue of the *Duke Mathematical Journal.*

[2] See, for example, T. Carleman, *Fonctions Quasi Analytiques*, Paris (1926), p. 12.

ON GREEN'S FORMULA

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Communicated October 11, 1940

Green's formula can be derived in the following way: For rectangular domains as the limit of an identity, for more complicated domains by virtue of elementary continuity properties of the integrals.

1. Rectangular Domains.—Let $u(x, y)$ be a function defined in the rectangle $R(a \leq x \leq A, b \leq y \leq B)$. If $a = x_0 < x_1 < \ldots < x_m = A$ and $b = y_0 < y_1 < \ldots < y_n = B$, then the following identity may be verified at a glance

$$\sum_{ij} \frac{u(x_i, y_{j+1}) - u(x_i, y_j)}{y_{j+1} - y_j} (x_{i+1} - x_i)(y_{j+1} - y_j) = \sum_i [u(x_i, B) - u(x_i, b)](x_{i+1} - x_i). \quad \text{(I)}$$

If $u_y = \dfrac{\partial u}{\partial y}$ exists in R, then by the mean value theorem

$$\frac{u(x_i, y_{j+1}) - u(x_i, y_j)}{y_{j+1} - y_j} = u_y(x_i, \eta_{ij})$$

where $y_j < \eta_{ij} < y_{j+1}$. Thus from (I) we get

$$\sum_{ij} u_y(x_i, \eta_{ij})(x_{i+1} - x_i)(y_{j+1} - y_j) = \sum_i [u(x_i, B) - u(x_i, b)](x_{i+1} - x_i). \quad \text{(II)}$$

If we assume that the Riemann integrals $\int\int_R u_y\, dx\, dy$ and $\int_a^A [u(x, B) - u(x, b)]dx$ exist, then in (II) the left side is a Riemann sum for the former, the right side one for the latter integral. By applying (II) to a sequence of decomposition of R whose norms approach 0, we thus get from (II)

$$\int\int_R u_y\, dx\, dy = \int_a^A [u(x, B) - u(x, b)]dx. \quad \text{(III)}$$

If we assume that $\int_a^A u(x, B)dx$ exists, then this integral is $=$

$$-\int_A^a u(x, B)dx,$$

and from (III) Green's formula for R follows:

$$\int\int_R u_y\, dx\, dy = -\int_R u(x, y)dx \quad \text{(IV)}$$

the integral on the right side being taken on the contour of R in the counter-clockwise sense.

2. Remarks and Corollaries.—Traditionally, Green's formula for simple domains is proved by iterated integration. But in order to apply the formula $\int\int_R f(x, y)dx\,dy = \int_a^A dx \int_b^B f(x, y)dy$ to $f(x, y) = u_y(x, y)$ one has to assume[1] that the function $F(x) = \int_b^B u_y(x, y)dy$ is integrable in $[a, A]$ while our proof of (III) does not require such an assumption. By means of Volterra's bounded function which is a derivative without admitting a Riemann integral it is easy to construct a function $u(x, y)$ for which $\int\int_R u_y\,dx\,dy$ and $\int_R u\,dx$ exist (thus our proof works) and yet $\int_b^B u_y(x, y)dy$, for some values of x, does not exist (thus the classical proof does not work).

In the same way as (II) we derive the identity (II′)

$$\sum_{ij} u_y(\xi_i, \eta_{ij})(x_{i+1} - x_i)(y_{j+1} - y_i) = \sum_i [u(\xi_i, B) - u(\xi_i, b)](x_{i+1} - x_i)$$

for any choice of ξ_i between x_i and x_{i+1}. If S is any Riemann sum for $\int_a^A [u(x, B) - u(x, b)]dx$ of a norm $<\nu$, then S is the right side of (II′) for a proper choice of the x_i and ξ_i. By (II′), S is equal to a Riemann sum for $\int\int_R u_y\,dx\,dy$ whose norm is $<\nu$. Thus from the existence of $\int\int_R u_y\,dx\,dy$ we conclude: $\int_a^A [u(x, B) - u(x, b)]dx$ and, in general, $\int_a^A [u(x, y') - u(x, y'')]dx$ for each y' and y'', exists. If $\int_a^A u(x, B)dx$ exists, then (IV) holds. If $\int_a^A u(x, y)dx$ exists for *one* value of y, it exists for *each* value of y. But $\int\int_R u_y\,dx\,dy$ may exist without $\int_a^A u(x, y)dx$ existing for any y. Example: $u(x, y) = f(x)$, for each y, where f is not Riemann integrable in $[a, A]$.

If we *define* the integral $\int\int_R u_y\,dx\,dy$ as the limit of the sums on the left side of (I) which may be called its Weierstrass sums,[2] then this integral exists and is equal to the Riemann integral $\int\int_R u_y\,dx\,dy$ whenever the latter exists. But the Weierstrass integral exists also in many other cases. Moreover, the Weierstrass integral automatically satisfies Green's formula whenever $\int_a^A [u(x, B) - u(x, b)]dx$ exists, i.e., whenever $u(x, B) - u(x, b)$ is almost everywhere continuous, without any assumption about the behavior of u in the interior of R.

3. Simple Closed Curves.—In a closed domain D let $u(p) = u(x, y)$ be continuous, $|u(p)| \leq U$, and $|u(p) - u(p')| < \sigma$ whenever the distance between $p = (x, y)$ and $p' = (x', y')$ is $<\delta(\sigma)$. We set $\lambda(p, p') = u(p)(x' - x)$ and $\lambda(\pi) = \sum\lambda(p_i, p_{i+1})$ if π is the ordered set $\{p_0, p_1, \ldots, p_n\}$. We remark that $|\lambda(\pi) - \lambda(p_0, p_n)| < \sigma l(\pi)$ if π is contained in a circle of diameter $\delta(\sigma)$, and $l(\pi)$ is the length of π. If C is a rectifiable curve, then we denote by $\lambda(C)$ the lim $\lambda(\pi)$ for the finite subsets of C, properly ordered, as they get indefinitely dense in C; that is to say, $\lambda(C) = \int_C u\,dx$. We have (1) $\lambda(-S) = -\lambda(S)$ if S and $-S$ are opposite segments; (2) $\lambda(V) =$

0 if V is a vertical segment; (3) $|\lambda(C) - \lambda(S)| < 2\sigma l(C)$ if the curve C, of length $l(C)$, from p to p' is contained in a circle of diameter $\delta(\sigma)$, and S is the segment pp'. For from the previous remark we have $|\lambda(C) - \lambda(p, p')| < \sigma l(C)$ and $|\lambda(S) - \lambda(p, p')| < \sigma l(S) \leq \sigma l(C)$. If in D the function u_y exists, is bounded, say, $|u_y| \leq U_y$, and admits a Riemann integral κ, then (4) $|\kappa(A)| \leq U_y a(A)$ for each domain A of area $a(A)$ for which $\kappa(A) = \int\int_A u_y \, dx \, dy$ exists.

If $T = \{p_0, p', p''\}$ is a triangle whose side $p'p''$ is vertical, then, by vertical segments, T may be decomposed into a triangle T_0 with the vertex p_0 and trapezoids $T_1, \ldots, T_n$ such that each T_i is the sum of a rectangle R_i and two triangles T_i' and T_i'' (or one if T has a horizontal side), all oriented in the same sense as T. In view of (2) and (3) we have $|\sum\lambda(T_i) - \sum\lambda(R_i)| < \sigma l(T)$ if the slices are so thin that each non-vertical side of the T_i is $< \delta(\sigma)$. Hence $|\lambda(T) - \sum\lambda(R_i)| < \sigma l(T) + Ul(T_0)$ which can be made as small as we please by choosing σ and T_0 sufficiently small. If, moreover, the slices are so thin that the area of $T_0 + \sum(T_i' + T_i'')$ is sufficiently small, then, in view of (4), $|\kappa(T) - \sum\kappa(R_i)|$ is arbitrarily small. Since Green's formula holds for each R_i it thus holds for T.

If P is a simple closed polygon, P can be decomposed into a finite number of triangles without common interior points, and each of them into at most two triangles with a common vertical side. Since Green's formula holds for each of these triangles, in view of (1) it holds for P.

If C: $p(t)$ $(0 \leq t \leq 1)$, $p(0) = p(1)$ is a closed rectifiable curve, we divide C into parts C_i between $p(t_i) = p_i$ and p_{i+1} $(0 = t_0 < t_1 < \ldots < t_n = 1)$ so small that each C_i is contained in a circle of diameter $\delta(\sigma)$. By adding the inequalities $|\lambda(C_i) - \lambda(S_i)| < 2\sigma l(C_i)$ resulting from (3), we get $|\lambda(C) - \lambda(P)| < 2\sigma l(C)$ where P is the polygonal line $\{p_0, p_1, \ldots, p_n\}$. Being rectifiable C can be covered with a finite number of squares whose sum, Q, has an arbitrarily small area. By choosing P so dense in C that it is contained in a Q whose area is sufficiently small, we make the area between C and P, and hence, by (4), $|\kappa(C) - \kappa(P)|$ as small as we please.[3] In view of Green's formula for P we conclude: If the simple closed rectifiable curve C admits simple closed inscribed polygons, arbitrarily dense in C, which together with their interior domains are contained in D, then Green's formula holds for C. In particular, the assumption is satisfied if D contains all points common to (1) the smallest convex set containing C, and (2) some open set containing C and its interior domain: e.g., if C and the interior domain of C are contained in the interior of D; or if C is convex and D is the domain bounded by C.

If D is a domain of the particular form $[a \leq x \leq A, \varphi(x) \leq y \leq \psi(x)]$ where φ and ψ are continuous in $[a, A]$, then without assuming any continuity properties of u we can prove by the method of sections 1 and 2

(i.e., by taking the limit of an identity): If in D the function u_y has a Riemann double integral, then

$$\int_a^A [u(x, \psi(x)) - u(x, \varphi(x))]dx$$

exists and is equal to the double integral. If $\int_a^A u(x, \omega(x))dx$ exists for *one* curve $y = \omega(x)$ $(a \leq x \leq A)$ contained in D, then the latter integral exists for *each* such curve. The integrability of u_y thus implies that the function u (if bounded) is almost everywhere continuous along each curve if it is so along one.

4. Closed Curves.—Let ρ be a point not on the closed curve C. We denote by $\alpha(\rho, t)$ the angle which the vector from ρ to $p(t)$ includes with the positive x-axis, and by $\mu_C(\rho)$ the integer $[\alpha(\rho, 1) - \alpha(\rho, 0)]/2\pi$. By a complementary domain of C we mean a component of the complementary set of C. For any two points of the same complementary domain U of C, μ_C has the same value which we call the multiplicity of U rel. C. The unbounded complementary domain of C has the multiplicity 0 rel. C.

If P is a closed polygon with the complementary domains $U_1, \ldots, U_n$ whose multiplicities rel. P are $\mu_1, \ldots, \mu_n$, respectively, then $\lambda(P) = \sum \mu_i \kappa(U_i)$, where it is sufficient to extend the sum over the complementary domains whose multiplicities rel. P are $\neq 0$. If P is a simple closed polygon, the statement is Green's formula. In the general case let P_i be the perimeter of U_i traversed in the counterclockwise sense. Since each P_i is a simple closed polygon, $\sum \mu_i \kappa(U_i) = \sum \mu_i \lambda(P_i)$. On the other hand, $\sum \mu_i \lambda(P_i) = \lambda(P)$. For let S be a segment of P common to P_i and P_j, ρ_i and ρ_j points of U_i and U_j close to the center of S. Then on $0 \leq t \leq 1$ the function $\alpha(\rho_i, t) - \alpha(\rho_j, t)$ increases by $2\pi(-2\pi)$ each time when S is traversed with U_i to the left (right), and only then. Thus in traversing P the algebraic number of times S is traversed with U_i to the left is $\mu_i - \mu_j$. But this is also the number of times S occurs in $\sum \mu_i \lambda(P_i)$. Segments bounding one domain only are easily seen to be traversed 0 times, all in all.

If C is a rectifiable curve with a finite number of complementary domains admitting inscribed polygons, arbitrarily dense in C, which together with their complementary domains of multiplicities $\neq 0$ lie in D, then

$$\int\int_C \mu(x, y) u_y(x, y)dx\, dy = \int_C u(x, y)dx.$$

Since for an inscribed polygon P the complementary domains approach those of C as P gets indefinitely dense in C, and $\lim \mu_P(\rho) = \mu_C(\rho)$ for each point ρ not on C, we get the theorem from that about closed polygons by the method used at the end of section 3.

5. Other Formulae of Vector Analysis.—The method of proving the formula for a rectangular domain as the limit of an identity, and for more

complicated domains by virtue of the elementary continuity properties of the integrals can be applied to many formulae of vector analysis. In this way E. Hemmingsen[4] proved $\int\int (u_x v_x + uv_{xx})dx\,dy = \int uv_x\,dy$. The Weierstrass sums of the left side contain second difference quotients. The identity underlying the rectangular case can be proved by an Abel transformation of the sum.

[1] Cf. de la Vallée Poussin, *Cours d'Analyse*, pp. 333–342.

[2] Bolza, *Lectures on the Calculus of Variations*, §31. A survey of the results obtained by considering *single* integrals as limits of their Weierstrass sums is contained in my paper, "Metric Geometry and Analysis," *Rice Pamphlets*, January, 1940.

[3] Cf. H. E. Bray, *Ann. Math.*, **26**, 278 (1925). We do not have to leave the domain bounded by C if we use (1) the geometric fact that there exists a simple closed polygon in the interior of C as close as we please to C, and such that the area of $C - P$ is arbitrarily small; (2) the continuity of $\lambda(C)$, for which an elementary proof can be derived from the author's more general theory (*Ergebn. e. math. Koll.*, **8**, 14–19 (1937).

[4] In his Master's thesis, University of Notre Dame, 1940.

SURFACES OF MINIMUM CAPACITY

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Communicated October 15, 1940

1. Capacities of Surface Caps.—We adopt the notation of an earlier note, in what follows.[1] In that note it was seen that, given a closed curve s in space, there exists a unique surface S bounded by it and, except in the neighborhood of s, composed of a finite number of smooth regular pieces, which satisfies the analog of the Euler condition for minimum capacity. Except for possible nodal lines and points this surface is analytic. It is assumed that s is of zero capacity and such that there exists a continuous one-one transformation ξ of a large sphere containing s into itself in such a way that each point of the surface corresponds to itself and the points of s correspond to the points of a circle σ interior to the sphere; for points outside the sphere ξ may be extended as the identical transformation.

THEOREM. *Of all the surfaces which may be considered as caps for s, the surface S has the minimum capacity.*

A surface S' is considered as a cap of s if (1) every point of s is a limit point of points of S', S' is bounded and S' is composed except in the neighborhood of s of a finite number of smooth regular pieces,[2] and (2) every closed curve which loops s contains a point of S'. We may assume that both sides of every regular surface element of S' are accessible from ∞, since otherwise we could diminish the capacity of S' by removing from

it a finite number of regular pieces of the element, and still retain the property (2).

With unit masses distributed on S and S', respectively, as conductor distributions (that is, with minimum energies) it has to be shown that the energy for S is greater than that for S'. These energies may be compared through the Dirichlet integrals. Let us suppose first that the surfaces are sufficiently smooth so that we may apply Green's theorem to the infinite regions Ω, Ω' complementary to them, and s sufficiently smooth so that every point of it is regular with respect to Ω and Ω'.

Denote by $\overline{V}(M)$, $\overline{U}(M)$ the conductor potentials of the conductor distributions of unit masses on S, S', respectively, and by $\bar{v}(M)$ the harmonic extension of $\overline{V}(M)$, according to the method of the cited paper,[1] as a function which is single valued on the two-leaved Riemann space $\mathfrak{M}$, harmonic except at points of s. The function $\bar{v}(M)$ is two valued at ∞, one branch taking on the value 0, the other the value $2/K(S)$ where $K(S)$ is the capacity of S. We have

$$D(\overline{V}) = \frac{4\pi}{K(S)}, \quad D(\overline{U}) = \frac{4\pi}{K(S')},$$

where $D(\overline{V})$, $D(\overline{U})$ are the corresponding Dirichlet integrals. For points on S we have $\overline{V} = \bar{v} = 1/K(S)$, and for points on S', $\overline{U} = 1/K(S')$.

2. *Application of Green's Theorem.*—The surface S' forms a cut surface for $\mathfrak{M}$ and $\bar{v}(M)$, since every closed curve which loops s contains a point of S'. Since S' is bounded, there is a single-valued branch of $\bar{v}$, which we denote by W, harmonic in Ω', and zero at ∞. We have

$$D(\overline{V}) = \int_{\Omega} (\nabla \overline{V})^2 dM = \frac{1}{2} \int_{\mathfrak{M}} (\nabla \bar{v})^2 dM = \int_{\Omega'} (\nabla W)^2 dM = D(W),$$

for the sum of the two values of $v(M)$ has everywhere the value $2/K(S)$.

Consider now $D(\overline{U}, W)$, the mixed integral $\int_{\Omega'} \nabla \overline{U} \cdot \nabla W \, dM$. By an application of Green's theorem we have

$$\int_{\Omega'} \nabla \overline{U} \cdot \nabla W \, dM = - \int_{S'} \overline{U} \frac{dW}{dn} dP,$$

where the integral of the second member is extended over the complete boundary of Ω', that is, so as to cover the two-sided local neighborhood of every point P of S'. But $\overline{U}$ is constant on S', and

$$- \int_{S'} \frac{dW}{dn} dP = - \int_{C} \frac{dW}{dn} dP = - \int_{C} \frac{d\overline{V}}{dn} dP = 4\pi,$$

where C is a sphere which includes the whole of S' in its interior. Consequently $D(\overline{U}, W) = 4\pi/K(S')$ and

$$D(\overline{U}, W - \overline{U}) = D(\overline{U}, W) - D(\overline{U}) = 0.$$

But now, writing $W = \overline{U} + (W - \overline{U})$, we have

$$D(\overline{V}) = D(W) = D(\overline{U}) + D(W - \overline{U}) + 2D(\overline{U}, W - \overline{U})$$

and

$$D(\overline{V}) = D(\overline{U}) + D(W - \overline{U}),$$

where $D(W - \overline{U}) \geq 0$.

3. *Proof of the Theorem.*—In order to treat the case where there may be points of s which are irregular boundary points of the domains complementary to S or S', we must consider an approximate situation in which s is replaced by a torus Σ_n surrounding it. Without restricting the generality assumed for s, it may be assumed that Σ_n is analytic and cuts S' in a finite number of regular arcs. As in the cited article, the approximate problem is solved for boundary values 1 on the torus, by means of a two-valued harmonic function v_n, and a surface S_n is defined as the locus $v_n = 1$. We take Σ_{n+1} inside Σ_n and let the Fréchet distance from Σ_n to s tend to zero as $n \to \infty$.

The conductor potential of the surface formed by Σ_n and the portion S_n' of S' outside Σ_n, corresponding to a conductor distribution of unit mass on $S_n' + \Sigma_n$ is denoted by $\overline{U}_n$, and functions $\overline{V}_n, \bar{v}_n, W_n$ are set up corresponding to unit mass on $S_n + \Sigma_n$, W_n being defined on the cut space Ω_n' outside $S_n' + \Sigma_n$. The function W_n may be taken as constant within Σ_n.

From the fact that v_n converges to v, it follows that the integral of the normal derivative for V_n (on any large sphere) converges to that for V, and $K(S_n + \Sigma_n)$ converges to $K(S)$, as $n \to \infty$. Hence

$$\lim_{n \to \infty} D(W_n) = D(W).$$

But also the sets $S_n' + \Sigma_n$ form a "decreasing" sequence with $S' + s$ as a limit, and

$$\lim_{n \to \infty} D(\overline{U}_n) = D(\overline{U}).$$

By writing

$$D(\overline{U}_n, W_n) - D(\overline{U}, W) = D(W_n - W, \overline{U}_n) + D(W, \overline{U}_n - \overline{U})$$

and

$$\frac{1}{4\pi} D(W_n - W, \overline{U}_n) = \int_{S_m'} (W_n - W)d\mu_n + \int_{\Sigma_{m,n}} (W_n - W)d\mu_n$$

$$\frac{1}{4\pi} D(W, \overline{U}_n - \overline{U}) = \int_{S_m'} Wd\mu_n - \int_{S_m'} Wd\mu + \int_{\Sigma_{m,n}} Wd[\mu_n - \mu],$$

where μ_n, μ are the respective distributions of unit mass on $S_n' + \Sigma_n$, $S' + s$, corresponding to the potentials $\overline{U}_n$, $\overline{U}$, and where $\Sigma_{m,n}$ denotes the portion of $S_n' + \Sigma_n$ inside Σ_m, we can make use of the weak convergence of μ_n to μ and the fact that s is of zero capacity to prove that, by fixing m great enough and then letting n become infinite, the two quantities $D(W_n - W, \overline{U}_n)$ and $D(W, \overline{U}_n - \overline{U})$ tend to zero. Consequently,

$$\lim_{n\to\infty} D(\overline{U}_n, W_n) = D(\overline{U}, W), \lim_{n\to\infty} D(W_n - \overline{U}_n) = D(W - \overline{U}).$$

Thence follows the relation

$$D(\overline{V}) = D(W) = D(\overline{U}) + D(W - \overline{U}).$$

If $D(W - \overline{U}) = 0$, it follows that $W \equiv \overline{U}$ except on $S' + s$, and that $D(W) = D(\overline{U})$; that is, $K(S) = K(S')$. Let P be a point not on $S' + s$. Then $W = \overline{U} < 1/K(S)$ and P is not on S; that is, S lies on $S' + s$. Let P' be a point not on $S + s$. Then at P', lim sup $W \neq 1/K(S)$ and lim sup $\overline{U} \neq 1/K(S')$; and P' is not on S'; that is, S' lies on $S + s$. Hence S and S' are identical.

Consequently, $D(\overline{V}) > D(\overline{U})$ and $K(S) > K(S')$ unless S and S' are identical.

In conclusion, I wish to express appreciation of the considerable assistance which has come through discussion of the problem with my colleague, Professor H. Lewy.

[1] G. C. Evans, "Surfaces of Minimal Capacity," these PROCEEDINGS, **26**, 489–491 (1940).

[2] "Smooth," for instance as in Kellogg, *Foundations of Potential Theory*, p. 157, Berlin (1929).

BOUNDARY VALUES OF FUNCTIONS SATISFYING A LINEAR PARTIAL DIFFERENTIAL EQUATION OF ELLIPTIC TYPE

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Communicated September 20, 1940

1. In the last few years some results and methods of the theory of analytic functions of one complex variable have been taken over for the study of functions $U(z, \bar{z})$,[1] satisfying a linear partial differential equation

$$\mathbf{L}(U) \equiv U_{z\bar{z}} + 2\mathrm{Re}[A(z, \bar{z})U_z] + C(z, \bar{z})U = 0,$$
$$U_z = {}^1/_2(\partial U/\partial x - i\partial U/\partial y),\ U_{\bar{z}} = {}^1/_2(\partial U/\partial x + i\partial U/\partial y) \quad (1.1)$$

where A is a complex and C is a real analytic function of two variables $z, \bar{z}$ regular in sufficiently large domain $|z| < \text{const.}, |\bar{z}| < \text{const.}$[2]

These investigations are based on the following

THEOREM 1. *For every* $\mathbf{L}$ *there exists a function* $\mathbf{E}(z, \bar{z};\ t) = 1 + z\bar{z}t^2\mathbf{E}^*(z, \bar{z};\ t)$ *such that every solution,* U, *of* $\mathbf{L}(U) = 0$ *regular in a star domain* $\mathfrak{F}^2$ *(of the xy-plane) can be represented in* $\mathfrak{F}^2$ *in the form*

$$U(z, \bar{z}) = \mathrm{Re}\left[\int_{-1}^{1} \mathbf{E}(z, \bar{z};\ t) f[{}^1/_2 z(1 - t^2)](1 - t^2)^{-1/2} dt\right] \quad (1.2)$$

where

$$f(\zeta) = \frac{2}{\pi}\int_0^{\pi/2} \zeta \sin\vartheta \frac{dU(2\zeta \sin^2\vartheta, 0)}{d(\zeta \sin^2\vartheta)}\, d\vartheta \quad (1.3)$$

(the associate of U*) is an analytic function of one complex variable regular in* $\mathfrak{F}^2$.

Conversely, the right-hand member of (1.2) *with* $f(\zeta)$ *regular in* $\mathfrak{F}^2$ *always represents a particular solution of* $\mathbf{L}(U) = 0$ *regular in* $\mathfrak{F}^2$.[3]

2. We will now prove the following result, which is of the Fatou[4] type.

THEOREM 2. *Let* $f(\zeta) = \sum_{n=0}^{\infty} a_n\zeta^n$ *and let*

$$\sum_{n=1}^{\infty} |a_n|^2 n^{-1} < \infty. \quad (2.1)$$

Then the function U *given by* (1.2) *has boundary values almost everywhere on the circle* $|z| = 2$ *for radial approach.*

Proof.—1°. By (2.1) and by Fatou's theorem for analytic functions of one complex variable, the function

$$g(z) = \int_{-1}^{1} f[{}^1/_2 z(1 - t^2)](1 - t^2)^{-1/2} dt = \sum_{n=0}^{\infty} z^n a_n 2^{-n} \int_{-1}^{1} (1 - t^2)^{n-1/2}\, dt =$$
$$\sum_{n=0}^{\infty} b_n z^n \quad (2.2)$$

has boundary values almost everywhere on the circle $|z| = 2$ for radial approach since[5]

$$|2^n b_n| \leqq |a_n| \int_{-1}^{1} (1 - t^2)^{n - 1/2} dt = |a_n| \Gamma(n + 1/2)\Gamma(1/2)/\Gamma(n + 1) \leqq c_1 |a_n|/\sqrt{n}.$$

2°. Suppose that $z = 2$ is a convergence point of (2.2). Then (as we will prove below) for every $\eta > 0$ and for every $\epsilon > \sigma$ there exists a $\delta(\eta,\epsilon)$, $\lim_{\epsilon \to 0} \delta(\eta,\epsilon) = 0$ (η fixed), so that

$$\left| \int_{-\eta}^{\eta} \left[\sum_{k=1}^{2} (-1)^k f[1/2\, r_k(1 - t^2)] \right] (1 - t^2)^{-1/2} dt \right| \leqq \epsilon \qquad (2.3)$$

for $|r_1 - r_2| \leqq \delta$, $0 \leqq r_k < 2$.

Since $g(r)$ is continuous in the closed interval $0 \leqq r \leqq 2$ (and therefore uniformly continuous) and since $f(\zeta)$ is regular for $|\zeta| \leqq (1 - \eta^2)$, $\eta > 0$, η fixed, our assertion follows from

$$\int_{-\eta}^{\eta} \left[\sum_{k=1}^{2} (-1)^k f[1/2\, r_k(1 - t^2)] \right] (1 - t^2)^{-1/2} dt = [g(r_2) - g(r_1)] - \left[\int_{-1}^{-\eta} + \int_{\eta}^{1} \right] \left\{ \left[\sum_{k=1}^{2} (-1)^k f[1/2\, r_k(1 - t^2)] \right] (1 - t^2)^{-1/2} dt \right\} \qquad (2.4)$$

3°. We proceed to the proof of the existence of $\lim_{r \to 2} U(r, r)$. By (1.2) we have

$$\left| \sum_{k=1}^{2} (-1)^k U(r_k, r_k) \right| \leqq \left| \left[\int_{-1}^{-\eta} + \int_{\eta}^{1} \right] \times \left\{ \left[\sum_{k=1}^{2} (-1)^k \mathbf{E}(r_k, r_k; t) f[1/2\, r_k (1 - t^2)] \right] (1 - t^2)^{-1/2} dt \right\} \right| +$$

$$+ \left| \int_{-\eta}^{\eta} \left[\sum_{k=1}^{2} (-1)^k f[1/2\, r_k(1 - t^2)] \right] (1 - t^2)^{-1/2} dt \right| + \qquad (2.5)$$

$$+ c_2 \sum_{k=1}^{2} \left| \int_{-\eta}^{\eta} t^2 f[1/2\, r_k(1 - t^2)] (1 - t^2)^{-1/2} dt \right| =$$

$$= |I_1(r_1, r_2)| + |I_2(r_1, r_2)| + |I_3(r,)| + |I_4(r_2)|, \; c_2 = 4 \max_{\substack{0 \leqq r \leqq 2 \\ 0 \leqq t \leqq 1}} |\mathbf{E}^*(r, r; t)|.$$

Let ϵ be an arbitrarily small positive number. We shall show that we can choose a positive η so small that each $|I_{2+k}(r_k)|$, $k = 1, 2$ becomes less than $\epsilon/4$.

Since $|f(\zeta)| \leqq \left[\sum_{n=1}^{\infty}|a_n|^2 n^{-1}\cdot\sum_{n=1}^{\infty} n|\zeta|^{2n}\right]^{1/2} \leqq c_3\,(1 - |\zeta|^2)^{-1}$ we have $|f[{}^1/_2 r_k(1-t^2)]| \leqq c_3\left[1 - [{}^1/_2 r_k(1-t^2)]^2\right]^{-1} \leqq c_4 t^{-2}$, $r_k < 2$, and therefore $|I_{2+k}(r_k)| \leqq 2c_2c_4\eta$.

After choosing $\eta < \epsilon/8c_2c_4$ we show in a way analogous to that of 2° that $|I_1(r_1, r_2)|$ can be made $< \epsilon/4$. Furthermore it follows by 2° that $|I_2(r_1, r_2)|$ can be made less than $\epsilon/4$. (In both cases we suppose that $|r_1 - r_2|$ is sufficiently small.)

Since by 1° the convergence points of $g(z)$ lie almost everywhere on $|z| = 2$, our theorem is proved.

3. Theorem 2 gives a sufficient condition for the existence of radial boundary values almost everywhere in terms of a property of the associate $f(\zeta)$ of $U(z, \bar{z}) = U(x + iy, x - iy) = W(x, y)$.

There arises the question of finding a method which permits us to decide whether (2.1) is fulfilled for f, $W(x, y)$ being given for *real* values of x, y only. In order to use (1.3) directly one must know the values of $U(z, \bar{z})$ also if z and $\bar{z}$ are not conjugate, i.e., of $W(x, y)$ if x and y are complex.

But by a slight modification of the above considerations we get a criterion for sufficiency. For every $\mathbf{L}$ there exist solutions $V(\zeta, \bar{\zeta}; z, \bar{z}) = V_1(\zeta, \bar{\zeta}; z, \bar{z}) \log|z - \zeta| + V_2(\zeta, \bar{\zeta}; z, \bar{z})$ of the adjoint equation $\mathbf{M}(V) = 0$, $\zeta \neq z$; V_1 and V_2 being regular functions of ζ, $\bar{\zeta}$ in a sufficiently large domain.[6] Using the known formula[7] we get the representation

$$U(z, \bar{z}) = {}^1/_2\pi^{-1}\int_0^{2\pi}[\tilde{U}(\rho, \varphi)K_1(\rho, \varphi; z, \bar{z}) + \tilde{U}_\rho(\rho, \varphi)K_2(\rho, \varphi; z, \bar{z})]d\varphi,$$
$$|z| < \rho, |\bar{z}| < \rho < 2,\ \tilde{U}(\rho, \varphi) = U(\rho e^{i\varphi}, \rho e^{-i\varphi}),\ \tilde{U}_\rho(\rho, \varphi) = \partial\tilde{U}(\rho, \varphi)/\partial\rho, \quad (3.1)$$

which is valid also if z and $\bar{z}$ are not conjugate; K_1, K_2 being certain functions connected in a simple manner with V.

By (1.3) and (3.1) we get

$$a_n = 1/2\,\pi^{-1}\int_0^{2\pi}[\tilde{U}(\rho, \varphi)K_1^{(n)}(\rho, \varphi) + \tilde{U}_\rho(\rho, \varphi)K_2^{(n)}(\rho, \varphi)]d\varphi, \qquad n > 0 \quad (3.2)$$

where $K_p^{(n)}$ are given by

$$\sum_{n=0}^{\infty}K_p^{(n)}(\rho, \varphi)\zeta^n = \frac{2}{\pi}\int_0^{\pi/2}\zeta\sin\vartheta\,\frac{dK_p(\rho, \varphi; 2\zeta\sin^2\vartheta, 0)}{d(\zeta\sin^2\vartheta)}\,d\vartheta,\ p = 1,2. \quad (3.3)$$

4. We can obviously use instead of V Green's function of $\mathbf{L}$ for the circle of radius ρ. Considerations like those of section 3 yield an analogous

expression for a_n, where $\tilde{U}(\rho, \varphi)$ alone appears. For applications, (3.2) is preferable since the properties of Green's function mentioned above are not very well known, while the behavior of the functions K_p used are comparatively simple.

However in some special cases it is possible to obtain for a_n expressions where only $\tilde{U}(\rho, \varphi)$ appears, without using the Green's function of $\mathbf{L}$.

We suppose now that $A = 0$ and C is a function which depends on r, but not on φ. One sees easily that $\mathbf{E}(z, \bar{z}; t)$ also depends only on r, and therefore we have for $U(z, \bar{z})$ the representation

$$U(r, \varphi) = \mathrm{Re}\left[\sum_{n=0}^{\infty} a_n e^{in\varphi} r^n \int_{-1}^{1} \tilde{\mathbf{E}}(r; t) \frac{(1 - t^2)^{n-1/2}}{2^n} dt\right],$$

$$\tilde{\mathbf{E}}(r; t) = \mathbf{E}(re^{i\varphi}, re^{-i\varphi}; t) \tag{4.1}$$

We get therefore

$$a_n = \frac{\int_0^{2\pi} \tilde{U}(\rho, \varphi) e^{-in\varphi} d\varphi}{\mathrm{C}_n(\rho)}, \quad \mathrm{C}_n(\rho) = \rho^n \int_{-1}^{1} \tilde{\mathbf{E}}(\rho; t) \frac{(1 - t^2)^{n-1/2}}{2^n} dt. \tag{4.2}$$

We remark that it can be easily shown[8] that

$$\mathrm{C}_n(\rho) \geqq c_5 \rho^n 2^{-n} n^{-1/2}, \; c_5 > 0 \tag{4.3}$$

for n sufficiently large. We notice furthermore that our procedure leads to various other theorems of Fatou's type for functions $U(z, \bar{z})$.

[1] We denote analytic functions of one complex variable z or ζ by small letters, analytic functions of two variables $z, \bar{z}$ or $\zeta, \bar{\zeta}$, respectively, by capital letters; $z = x + iy$, $\bar{z} = x - iy$, $\zeta = \xi + i\eta$, $\zeta = \xi - i\eta$; x, y and ξ, η, respectively, being Cartesian coördinates.

[2] See the following papers: [1] Bergman, Stefan, *Recueil mathématique*, **2** (44) (new series), 1169–1198 (1937); [2] Bergman, Stefan, *Duke Math. Jour.*, **6**, 537–561 (1940).

We suppose that A and C can be analytically prolonged in a certain domain of the complex variables x and y. (Notice that for x and y complex, z and $\bar{z}$ are not longer conjugate.)

[3] See Bergman, *Recueil mathématique*, **2** (44), (new series) 1172–1179 (1937).

[4] We notice that criteria for the existence of boundary values almost everywhere are obtained in some cases immediately from an analogous theorem for subharmonic functions. (Cf. Privaloff, I. I., and Kouznetzoff, P., *Recueil mathématique*, **6** (48) (new series), 345–376 (1939).) If we suppose that U is non-negative in the unit circle and $A = 0$, then we can divide the circle into a finite number of regions so that in every region U is either super- or subharmonic and we then use the theorem cited.

[5] See Nielsen, N., *Handbuch der Theorie der Gammafunktionen*, Leipzig, 1906, p. 92. We denote by c_k appropriate constants.

[6] See Hedrick, R., *Ueber den analytischen Charakter der Lösungen von Differentialgleichungen*, Dissertation, Göttingen, 1901.

[7] See Sommerfeld, A., *Enzykl. der math. Wissensch.*, **IIA**, 7*c*, 504–570 (1900–1916), especially p. 515.

[8] See Bergman, *Recueil mathématique*, **2** (44) (new series), 1181 (1937).